MARKOV PROCESSES WHOSE HITTING DISTRIBUTIONS ARE DOMINATED BY THOSE OF A GIVEN PROCESS(1)

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Introduction. Let X be a Markov process. Let X^* be the process obtained from a random time change in a subprocess of X. Then obviously the hitting distributions of X^* are dominated by those of X. It has been naturally conjectured that the converse is true under broad conditions. The exact statement of the converse would be as follows: if two Markov processes are such that the hitting distributions of X dominate those of X^*, then there is a process Y obtained from a random time change in a subprocess of X that is equivalent to X^*. The conjecture is proved by Šur [6] in case X is a Brownian motion process and X^* is a standard process. This paper deals with the very general case where X and X^* are Hunt processes with a locally compact separable metric space as their common state space. (Our definition of a Hunt process agrees essentially with that of a standard process.) The conjecture is proved true under a trivially necessary condition, provided a slight change in the definition of a subprocess is allowed (see §1). Our method is to construct a multiplicative functional of X which induces a process with the same hitting distributions as X^*. When the state space of X^* is a proper subspace of that of X which is a locally compact separable metric space, it seems that no difficulty would arise in obtaining a similar result with the same techniques.

1. **Preliminaries and the main result.** Let E be a locally compact separable metric space. Let $E_\Delta = E \cup \{\Delta\}$ be the one-point compactification of E. Denote by $\mathscr{B}$ and $\mathscr{B}_\Delta$ respectively the Borel fields of E and E_Δ. A Hunt process X with E as its state space is a structure $(\Omega, \mathscr{M}, X(t), P^x, \theta_t)$ where Ω is a set (the sample space), $\mathscr{M}$ a σ-field on Ω, $X(t)$, $0 \leqq t < \infty$, are random variables on $(\Omega, \mathscr{M})$ taking values in E_Δ, P^x, $x \in E_\Delta$, probability measures on $\mathscr{M}$ with $P^x(X(0)=x)=1$, and θ_t, $0 \leqq t < \infty$, operators on Ω satisfying $X(s)(\theta_t) = X(s+t)$. For each $\omega \in \Omega$, the path function $t \to X_t(\omega) = X(t, \omega) = X(t)(\omega)$ is right continuous and has a left limit at every t, and $X(t, \omega) = \Delta$ if $t \geqq \sigma(\omega) = \inf\{s \mid X(s) = \Delta\}$, the lifetime of ω. Let $\mathscr{G}_t$ be the σ-field generated by $X(s)$, $s \leqq t$ and $\mathscr{G}$ that generated by $X(t)$, $t < \infty$. For each

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$\Lambda \in \mathscr{G}$, $x \to P^x(\Lambda)$ is $\mathscr{B}_\Delta$-measurable. For a probability measure μ on $\mathscr{B}_\Delta$ (μ will always denote such a measure) P^μ is defined on $\mathscr{G}$ by $P^\mu(\Lambda) = \int P^x(\Lambda)\mu(dx)$. Let $\mathscr{G}^\mu$ be the P^μ-completion of $\mathscr{G}$, and let $\mathscr{F}$ be the intersection of $\mathscr{G}^\mu$ over all μ. Let $\mathscr{F}_t$ be the σ-field of sets Λ satisfying the condition that, for each μ, there exist $\Lambda_1 \in \mathscr{G}_t$ and $\Lambda_2 \in \mathscr{G}$ with $P^\mu(\Lambda_2) = 0$ and $(\Lambda - \Lambda_1) \cup (\Lambda_1 - \Lambda) \subset \Lambda_2$. A stopping time T is a function from Ω to $[0, \infty]$ such that $\{T < t\} \in \mathscr{F}_t$ for every t. For a stopping time T, $\mathscr{F}_T = \mathscr{F}(T)$ denotes the σ-field of sets $\Lambda \in \mathscr{F}$ satisfying $\Lambda \cap \{T < t\} \in \mathscr{F}_t$ for every t. If ϕ is a bounded real-valued $\mathscr{F}$-measurable function and T a stopping time, then $E^x\{\phi(\theta_T); \Lambda \cap [T < \infty]\} = E^x\{E^{X(T)}(\phi); \Lambda \cap [T < \infty]\}$ for all $\Lambda \in \mathscr{F}(T)$. This is the strong Markov property of X. If T_n are stopping times increasing to T, then $X(T_n)$ converge to $X(T)$ a.e. on $\{T < \sigma\}$, where a.e. means a.e. P^x for every x. This is called the quasi-left continuity of X. Note that it is not required that X satisfies the stronger form of quasi-left continuity that the above convergence holds a.e. on $\{T < \infty\}$. Thus our definition of a Hunt process agrees with that of a standard process.

For an analytic subset A of E and $x \in E$, the hitting distribution $H_A(x, \cdot)$ is defined on $\mathscr{B}$ by $H_A(x, B) = P^x(X(T_A) \in B, T_A < \infty)$, where $T_A = \inf\{t > 0 \mid X(t) \in A\}$ is the hitting time of A. A point x is regular for A if $P^x(T_A = 0) = 1$, and irregular if otherwise. A point $x \in E$ is a holding point if it is irregular for $E - \{x\}$. The set of holding points will be denoted by H. The points in $I = E - H$ are called instantaneous points. A holding point is called a trap if $H_{E-\{x\}}(x, E) = 0$. Let T be a stopping time $\leqq \sigma$ such that $T(\theta_t) = T - t$ on $\{t < T\}$. Let $Z(t) = X(t)$ on $\{t < T\}$ and $= \Delta$ elsewhere. Then $Z = (\Omega, \mathscr{M}, Z(t), P^x, \theta_t)$ is again a Hunt process, whose state space is $E' = \{x \in E \mid P^x(T > 0) = 1\}$. (If X is only a strong Markov process then so is Z.) Such a process is called a subprocess of X. A multiplicative functional M of X is a real-valued function on $[0, \infty) \times \Omega$ such that (i) $t \to M(t)(\omega) = M(t, \omega)$ is right continuous, nonincreasing and taking values in $[0, 1]$ for almost all ω, (ii) $M(t)$ is $\mathscr{F}(t)$-measurable, and (iii) $M(s+t) = M(t)[M(s)(\theta_t)]$ a.e. If M is a multiplicative functional such that $M(t) = 0$ on $\{\sigma \leqq t\}$, then one can define a Markov process Y in the manner of [5, p. 142] from X and M. Y has $E' = \{x \in E \mid P^x(M(0) = 1) = 1\}$ as its state space. The transition function $q(t, x, B)$, $x \in E'$, $B \in \mathscr{B} \cap E'$, of Y is given by $E^x\{M(t); X(t) \in B\}$ and its hitting distribution $H_A^Y(x, B)$, $x \in E$, $A \subset E'$, is equal to $E^x\{M(T_{A'}); X(T_{A'}) \in B, T_{A'} < \infty\}$ where $A' = (E - E') \cup A$. Such a process we will also call a subprocess of X.

For a summary of the basic definitions and facts of the theory of Hunt processes we refer the reader to §1 of [4] or §2 of [1]. A discussion of multiplicative functionals can be found in [3], [6] or Chapter 10 of [2].

Let us consider two Hunt processes X and X^* with E as their common state space. For notational convenience suppose X and X^* are defined on the same sample space $(\Omega, \mathscr{M})$ and by the same random variables $X(t)$. Thus we write $X^* = (\Omega, \mathscr{M}, X(t), {}^*P^x, \theta_t)$. Let ${}^*P^\mu$, $\mathscr{F}^*$, $\mathscr{F}_t^*$, $\mathscr{F}_T^*$, $H_A^*(x, B)$, H^* and I^* be defined from the probability measures ${}^*P^x$ in the same manner as P^μ, $\mathscr{F}$, etc. are defined from P^x; a.e. * will mean a.e. $*P^x$ for every x. If $\mathcal{S}$ is sub-σ-field of $\mathcal{F}$ ($\mathcal{F}^*$), $\mathcal{S}^\mu$ ($\mathcal{S}^{*\mu}$) will denote the σ-field of sets Λ such that there exist $\Lambda_1 \in \mathcal{S}$ and $\Lambda_2 \in \mathcal{F}$ ($\Lambda_2 \in \mathcal{F}^*$) with $P^\mu(\Lambda_2)=0$ ($*P^\mu(\Lambda_2)=0$) and $(\Lambda-\Lambda_1) \cup (\Lambda_1-\Lambda) \subset \Lambda_2$. Note that this definition agrees with a previous one if $\mathcal{S}=\mathcal{G}$. For $\Lambda \subset \Omega$, $\mathcal{S} \cap \Lambda = \{\Lambda_1 \cap \Lambda \mid \Lambda_1 \in \mathcal{S}\}$. Let $\mathcal{C}_0$ be the set of real continuous functions on E vanishing at infinity.

Our basic hypotheses are:

A. $H_K^*(x, \cdot) \leqq H_K(x, \cdot)$ for every compact subset K of E and $x \in E$.

B. If x is a trap for X^*, then $x \in H$.

The main result is the following

THEOREM. *Under hypotheses* A *and* B *there exists a multiplicative functional M of X such that $E^x\{M(T_A); X(T_A) \in B, T_A < \infty\} = H_A^*(x, B)$ for all $x \in E$, analytic subsets A of E and $B \in \mathcal{B}$.*

Obviously the multiplicative functional M in the theorem must satisfy $M(0)=1$ a.e. P^x for all $x \in E$, and we may assume $M(t)=0$ on $\{\sigma \leqq t\}$. Thus the subprocess Y of X induced by M has E as its state space and has the same hitting distributions (on analytic subsets of E) as X^*. One can then appeal to the result of [1] to obtain a random time change in Y which induces a process equivalent to X^*.

A remark about the hypotheses: if the result of the theorem holds and if $x \in I$, then x is not a trap for X^*, since

$$H_{E-\{x\}}^*(x, E) = E^x\{M(T_{E-\{x\}}); X(T_{E-\{x\}}) \in E, T_{E-\{x\}} < \infty\} = 1.$$

Thus hypothesis B is necessary. Hypothesis A implies that $H_A^*(x, \cdot) \leqq H_A(x, \cdot)$ for all analytic subsets A of E. This is a result of the fact that, in a Hunt process, the hitting time of an analytic set can be approximated by the hitting time of its compact subsets.

Let us choose a metric ρ on E_Δ. Let reg A (reg* A) be the set of regular points for A relative to X (X^*). We will now prove two immediate results of hypotheses A and B.

PROPOSITION 1.1. $H=H^*$.

Proof. If $x \in H$, then $H_{E-\{x\}}^*(x, \{x\}) \leqq H_{E-\{x\}}(x, \{x\})=0$, and hence x cannot be in I^*. So $H \subset H^*$. Suppose $x \in H^*$. If $H_{E-\{x\}}^*(x, E-\{x\})=0$, then x is a trap for X^* and so $x \in H$ by hypothesis B. If $H_{E-\{x\}}^*(x, E-\{x\})>0$, then $H_{E-\{x\}}(x, E-\{x\})>0$ and so $x \notin I$. So $H^* \subset H$.

PROPOSITION 1.2. *If K is a compact subset of E, then* reg $K=$ reg* K.

Proof. If $x \in K$, then trivially $x \notin$ reg K and also $x \notin$ reg* K. Suppose $x \in K$. If $x \in H=H^*$, then of course $x \in$ reg $K \cap$ reg* K. Suppose $x \in I=I^*$. Let

$$D_n = K \cup \{y \mid 1/n \leqq \rho(x, y) \leqq \rho(x, \Delta)/2\}.$$

Then we clearly have: $H_{D_n}(x, \{x\})=1$ for all n if and only if $x \in \text{reg } K$, $H^*_{D_n}(x, \{x\})=1$ for all n if and only if $x \in \text{reg}^* K$ and $H^*_{D_n}(x, D_n) \uparrow 1$ as $n \to \infty$. These conditions and hypothesis A imply that $x \in \text{reg } K$ if and only if $x \in \text{reg}^* K$. The proof is complete.

2. **Partitioning of the state space and a family of stopping times.** A pair $(\mathscr{U}, \mathscr{V})$ will be called a partition of E if $\mathscr{U}=\{U_1, \ldots, U_p\}$ is a finite covering of E by open subsets of E, and $\mathscr{V}=\{V_1, \ldots, V_p\}$ is an (ordinary) partition of E by Borel sets with $V_i \subset U_i$ for all i. A partition $(\mathscr{U}, \mathscr{V})$ of E is a refinement of another partition $(\mathscr{U}', \mathscr{V}')$ if, whenever $V_i \in \mathscr{V}$, $U_j' \in \mathscr{U}'$, and $V_i \cap U_j' \neq \varnothing$, we have $U_i \subset U_j'$. It is easily checked that being a refinement of is a transitive relation among partitions $(\mathscr{U}, \mathscr{V})$ of E. For a family $\mathscr{U}$ of subsets of E let $|\mathscr{U}|$ be the sup over $U \in \mathscr{U}$ of diam $U=\sup\{\rho(x, y) \mid x, y \in U\}$. The following observation is basic to our study.

PROPOSITION 2.1. *There exists a sequence* $(\mathscr{U}^{(n)}, \mathscr{V}^{(n)})$, $n=1, 2, \ldots$, *of partitions of E such that* (i) $|\mathscr{U}^{(n)}|<1/n$ *for all* n, (ii) $(\mathscr{U}^{(n)}, \mathscr{V}^{(n)})$ *is a refinement of* $(\mathscr{U}^{(m)}, \mathscr{V}^{(m)})$ *if* $m \leqq n$, *and* (iii) $V_i^{(n)}=U_i^{(n)}-\bigcup_{j=1}^{i-1} U_j^{(n)}$ *for all n and* $i>1$.

Proof. For each n choose a finite covering $\mathscr{W}^{(n)}$ of E consisting of open subsets of E such that $|\mathscr{W}^{(n)}|<1/n$ (note that ρ is a metric on E_Δ) and $\mathscr{W}^{(n)}$ is closed w.r.t. finite intersection. Define $\mathscr{U}^{(n)}$ inductively by setting $\mathscr{U}^{(1)}=\mathscr{W}^{(1)}$ and $\mathscr{U}^{(n+1)}=\{U_1 \cap U_2 \mid U_1 \in \mathscr{U}^{(n)} \text{ and } U_2 \in \mathscr{W}^{(n+1)}\}$ for $n \geqq 1$. Clearly each $\mathscr{U}^{(n)}$ is closed w.r.t. finite intersection. Enumerate numbers of $\mathscr{U}^{(n)}$ as $U_1^{(n)}, \ldots, U_p^{(n)}$ in such a manner that $i \leqq j$ whenever $U_i^{(n)} \subset U_j^{(n)}$. Let $V_1^{(n)}=U_1^{(n)}$ and $V_i^{(n)}=U_i^{(n)}-\bigcup_{j=1}^{i-1} U_j^{(n)}$ for $i>1$. (i) and (iii) are immediate. We now show $(\mathscr{U}^{(n+1)}, \mathscr{V}^{(n+1)})$ refines $(\mathscr{U}^{(n)}, \mathscr{V}^{(n)})$. Suppose $x \in V_i^{(n+1)} \cap U_j^{(n)}$. We need to show $U_i^{(n+1)} \subset U_j^{(n)}$. It is clear that $U_i^{(n+1)}=\bigcap\{U \in \mathscr{U}^{(n+1)} \mid x \in U\}$. On the other hand $U_i^{(n+1)}=U_1 \cap U_2$ for some $U_1 \in \mathscr{U}^{(n)}$ and $U_2 \in \mathscr{W}^{(n+1)}$. Since $x \in U_j^{(n)} \cap U_2 \in \mathscr{U}^{(n+1)}$, we have $U_i^{(n+1)} \subset U_j^{(n)} \cap U_2 \subset U_j^{(n)}$. This proves $(\mathscr{U}^{(n+1)}, \mathscr{V}^{(n+1)})$ is a refinement of $(\mathscr{U}^{(n)}, \mathscr{V}^{(n)})$, and it follows from transitivity that $(\mathscr{U}^{(n)}, \mathscr{V}^{(n)})$ refines $(\mathscr{U}^{(m)}, \mathscr{V}^{(m)})$ if $m<n$. It remains to show $(\mathscr{U}^{(n)}, \mathscr{V}^{(n)})$ is a refinement of itself. But this is obvious since if $x \in V_i^{(n)} \cap U_j^{(n)}$, then $U_i^{(n)} = \bigcap\{U \in \mathscr{U}^{(n)} \mid x \in U\} \subset U_j^{(n)}$. We choose a sequence $\{(\mathscr{U}^{(n)}, \mathscr{V}^{(n)})\}$ of partitions of E satisfying the conditions in the above proposition. Let us define a family of stopping times as follows. First, for each n let $T^{(n)}=T_{E-U_i^{(n)}}$ if $X(0) \in V_i^{(n)}$, and $=\infty$ if $X(0)=\Delta$. It is clear that each $T^{(n)}$ is a stopping time (for both processes X and X^*—as is every stopping time to be defined in the sequel). Let π be the first uncountable ordinal. For each n define inductively stopping times $T_\alpha^{(n)}$, $\alpha<\pi$, as follows:

$$T_0^{(n)} = 0;\ T_{\alpha+1}^{(n)} = T_\alpha^{(n)}+T^{(n)}(\theta_{T_\alpha^{(n)}}) \quad \text{if } T_\alpha^{(n)} < \infty, \text{ and } =\infty \text{ if } T_\alpha^{(n)} = \infty;$$

$$T_\alpha^{(n)} = \sup_{\gamma<\alpha} T_\gamma^{(n)} \quad \text{if } \alpha \text{ is a limit ordinal.}$$

The right continuity of path functions implies that $T^{(n)}>0$ and hence $T_{\alpha+1}^{(n)}>T_\alpha^{(n)}$ provided $T_\alpha^{(n)}<\infty$. It follows that for each ω there is $\alpha<\pi$ such that $T_\alpha^{(n)}(\omega)=\infty$. Also, a standard argument shows that for each μ there exists $\beta<\pi$ such that $P^\mu(T_\beta^{(n)}<\infty)={}^*P^\mu(T_\beta^{(n)}<\infty)=0$.

PROPOSITION 2.2. *Let $m<n$ and $\omega \in \Omega$. If $X(0,\omega) \in U_i^{(m)}$, then, for some α, $T_\alpha^{(n)}(\omega)=T_{E-U_i^{(m)}}(\omega)$.*

Proof. We may assume $T_{E-U_i^{(m)}}(\omega)<\infty$. Then $\gamma_0=\sup\{\gamma \mid T_\gamma^{(n)}(\omega) \leqq T_{E-U_i^{(m)}}(\omega)\}<\pi$ and $T_{\gamma_0}^{(n)}(\omega) \leqq T_{E-U_i^{(m)}}(\omega)$. If the equality does not hold, then $x=X(T_{\gamma_0}^{(n)}(\omega),\omega) \in U_i^{(n)}$. (Note that $x=\Delta$ implies $T_{E-U_i^{(m)}}(\omega)=\infty$.) Suppose $x \in V_j^{(n)}$. Then since $(\mathscr{U}^{(n)}, \mathscr{V}^{(n)})$ is a refinement of $(\mathscr{U}^{(m)}, \mathscr{V}^{(m)})$ we have $U_j^{(n)} \subset U_i^{(m)}$. This implies $T^{(n)}(\theta_{T_{\gamma_0}^{(n)}}\omega)$ $\leqq T_{E-U_i^{(m)}}(\theta_{T_{\gamma_0}^{(n)}}\omega)$ and hence $T_{\gamma_0+1}^{(n)}(\omega) \leqq T_{E-U_i^{(m)}}(\omega)$, a contradiction.

COROLLARY 2.3. *Let $m<n$, $\alpha<\pi$, and $\omega \in \Omega$. There is $\lambda<\pi$ satisfying*

$$T_\lambda^{(n)}(\omega) = T_\alpha^{(m)}(\omega).$$

Proof. The case $\alpha=1$ follows immediately from the previous proposition and the definition of $T_1^{(m)}(\omega)$. Suppose the assertion is true if α is replaced by any smaller ordinal. If α has a predecessor γ and $T_\gamma^{(m)}(\omega)=T_{\lambda_1}^{(n)}(\omega)$, then assuming as we may that $T_\gamma^{(m)}(\omega)<\infty$, we can find λ_2 such that $T_{\lambda_2}^{(n)}(\theta_{T_{\lambda_1}^{(n)}}\omega)=T_{\lambda_2}^{(n)}(\theta_{T_\gamma^{(m)}}\omega)=T_1^{(m)}(\theta_{T_\gamma^{(m)}}\omega)$. Thus we have $T_{\lambda_1+\lambda_2}^{(n)}(\omega)=T_\alpha^{(m)}(\omega)$. If α is a limit ordinal, let $\lambda(\gamma)$ be such that $T_{\lambda(\gamma)}^{(n)}(\omega)$ $=T_\gamma^{(m)}(\omega)$ for $\gamma<\alpha$ and let $\lambda=\sup\{\lambda(\gamma) \mid \gamma<\alpha\}$. Then $T_\lambda^{(n)}(\omega)=T_\alpha^{(m)}(\omega)$ by definition. The corollary then follows from induction.

The fact asserted in the above corollary, which may be called the interpolation property of the $T_\alpha^{(n)}$, is fundamental to this work. Let $\lambda(m,\alpha,n)=\inf\{\lambda<\pi \mid T_\lambda^{(n)}=T_\alpha^{(m)}\}$ for $m \leqq n$. Note that $\lambda(m,\alpha,n)$ is a function from Ω into the set of countable ordinals, and $\{\lambda(m,\alpha,n)=\gamma\}=\{T_\alpha^{(m)}=T_\gamma^{(n)}>T_\delta^{(n)}>$ for all $\delta<\gamma\}$ is in $\mathscr{F}(T_\alpha^{(m)}) \cap \mathscr{F}(T_\gamma^{(n)})$ and also in $\mathscr{F}(T_\alpha^{(m)})^* \cap \mathscr{F}(T_\gamma^{(n)})^*$.

COROLLARY 2.4. *Let $m<n$. If $T_\alpha^{(m)}(\omega) \geqq t$, there exists $\gamma<\pi$ such that*

$$T_\alpha^{(m)}(\omega) = t+T_\gamma^{(n)}(\theta_t\omega).$$

Proof. The case $T_\alpha^{(m)}(\omega)=t$ is trivial and we may assume $T_\alpha^{(m)}(\omega)>t$. Let $\alpha_0=\sup\{\alpha_1 \mid T_{\alpha_1}^{(m)}(\omega) \leqq t\}$. Then $T_{\alpha_0}^{(m)}(\omega) \leqq t<T_{\alpha_0+1}^{(m)}(\omega)$. Suppose $X(T_{\alpha_0}^{(m)}(\omega),\omega) \in V_i^{(m)}$. Then $X(t,\omega) \in U_i^{(m)}$ and $T_{\alpha_0+1}^{(m)}-t$ is the first time $\theta_t\omega$ hits $E-U_i^{(m)}$. Now Proposition 2.2 implies that $T_{\alpha_0+1}^{(m)}(\omega)-t=T_\gamma^{(n)}(\theta_t\omega)$ for some γ. This proves the case when $\alpha=\alpha_0+1$. The case when $\alpha>\alpha_0+1$ is then easy to see.

For a compact subset K of E, we can define stopping times in terms of $T_\alpha^{(n)}$ to approximate T_K. Let $T_n=\inf\{T_\alpha^{(n)}<\sigma \mid$ if $X(T_\alpha^{(n)}) \in V_i^{(n)}$ then $U_i^{(n)} \cap K \neq \varnothing\}$ and $R_n=\inf\{T_\alpha^{(n)} \mid T_\alpha^{(n)} \geqq T_K\}$.

PROPOSITION 2.5. *Each T_n is a stopping time. If $x \notin K$, then $T_n \uparrow T_K$ or σ a.e. P^x and a.e. $^*P^x$ as $n \to \infty$.*

Proof. Let $F_n=\bigcup\{V_i^{(n)} \mid U_i^{(n)} \cap K \neq \varnothing\}$. From the fact that $(\mathscr{U}^{(n)}, \mathscr{V}^{(n)})$ refines itself one easily checks that $T_n=0$ if $X(0) \in F_n$ and $T_n=T_{F_n}$ if otherwise. Hence T_n is a stopping time. The rest follows from the fact that $\rho(y,K)=\inf\{\rho(y,z) \mid z \in K\}$ $<1/n$ for $y \in F_n$ and the quasi-left continuity.

PROPOSITION 2.6. *Each R_n is a stopping time. If $K \subset I$, then $R_n \downarrow T_K$ a.e. and a.e.**.

Proof. $\{R_n < t\} = \bigcup_{\alpha<\pi} \{R_n = T_\alpha^{(n)} < t\}$. Now $\{R_n = T_\alpha^{(n)}\} = \{T_\alpha^{(n)} \geqq T_K > T_\gamma^{(n)}$ for all $\gamma < \alpha\} \in \mathscr{F}(T_\alpha^{(n)})$. Hence $\{R_n = T_\alpha^{(n)} < t\} \in \mathscr{F}(t)$. From the fact that for each μ there exists β such that $P^\mu(T_\beta^{(n)} < \infty) = 0$ we see that $\{R_n < t\} \in \mathscr{F}(t)$. Similarly $\{R_n < t\} \in \mathscr{F}(t)^*$. Hence R_n is a stopping time. Suppose $K \subset I = I^*$. It follows from the strong Markov property that, a.e. (a.e.*) on $\{T_K < \infty\}$, there is an arbitrarily small $\delta > 0$ such that $X(T_K) \neq X(T_K + \delta)$. Clearly if $X(t, \omega) \neq X(t+\delta, \omega)$ then there exists some (n, α) such that $t < T_\alpha^{(n)}(\omega) \leqq t + \delta$. Hence $R_n \downarrow T_K$ a.e. and a.e.*.

3. **Some measurability lemmas.** For each positive integer n and countable ordinal γ, let $\mathscr{H}(n, \gamma)$ be the σ-field generated by sets of the form

$$\{X(T_\delta^{(n)}) \in B,\ T_\delta^{(n)} < \infty\}, \quad B \in B_\Delta,\ \delta \leqq \gamma.$$

Since $\{X(T_\delta^{(n)}) \in B, T_\delta^{(n)} < \infty\} \in \mathscr{F}(T_\delta^{(n)})$ and, for $\delta < \gamma$, $\mathscr{F}(T_\delta^{(n)}) \subset \mathscr{F}(T_\gamma^{(n)})$, $\mathscr{H}(n, \gamma)$ is a sub-σ-field of $\mathscr{F}(T_\gamma^{(n)})$. Also, $\mathscr{H}(n, \gamma) \subset \mathscr{H}(n, r')$ whenever $\gamma < \gamma'$.

LEMMA 3.1. *If $m \leqq n$, then $\{T_\alpha^{(m)} = T_\gamma^{(n)} < \infty\} \in \mathscr{H}(n, \gamma)$.*

Proof. We prove by induction on both γ and α. The induction hypothesis is that $\{T_{\gamma'}^{(n)} = T_{\alpha'}^{(m)} < \infty\} \in \mathscr{H}(n, \gamma')$ whenever $\gamma' < \gamma$ and $\alpha' < \pi$, or $\gamma' = \gamma$ and $\alpha' < \alpha$. Since the case $\alpha = 0$ is trivial we assume $\alpha > 0$. If α has a predecessor α', then

$$\begin{aligned}\{T_\gamma^{(n)} = T_\alpha^{(m)} < \infty\} = \bigcup_{\gamma'<\gamma} (\{T_{\gamma'}^{(n)} = T_{\alpha'}^{(n)} < \infty\} \cap \bigcup_i \{X(T_{\gamma'}^{(n)}) \in V_i^{(m)};\\ X(T_\delta^{(n)}) \in U_i^{(m)} \text{ for } \gamma' < \delta < \gamma;\ X(T_\delta^{(n)}) \in E - U_i^{(m)},\ T_\delta^{(n)} < \infty\})\\ \in \mathscr{H}(n, \gamma).\end{aligned}$$

If α has no predecessor, then $T_\alpha^{(m)} = \sup\{T_{\alpha'}^{(m)} \mid \alpha' < \alpha\}$. Now for $\alpha' < \pi$,

$$\{T_{\alpha'}^{(m)} < T_\gamma^{(n)} < \infty\} = \{T_\gamma^{(n)} < \infty\} \cap \bigcup_{\gamma'<\gamma} \{T_{\alpha'}^{(m)} = T_{\gamma'}^{(n)} < \infty\} \in \mathscr{H}(n, \gamma).$$

Hence if $\alpha' < \pi$, then

$$\{T_\gamma^{(n)} < T_{\alpha'}^{(m)}\} = \{T_\gamma^{(n)} < \infty\} - \{T_{\alpha'}^{(m)} = T_\gamma^{(n)} < \infty\} - \{T_{\alpha'}^{(m)} < T_\gamma^{(n)} < \infty\} \in \mathscr{H}(n, \gamma).$$

It follows that

$$\begin{aligned}\{T_\alpha^{(m)} = T_\gamma^{(n)} < \infty\} &= \{T_\gamma^{(n)} < \infty\} - \{T_\alpha^{(m)} < T_\gamma^{(n)} < \infty\} - \{T_\gamma^{(n)} < T_\alpha^{(m)}\}\\ &= \{T_\gamma^{(n)} < \infty\} - \{T_\alpha^{(m)} < T_\gamma^{(n)} < \infty\} - \bigcup_{\alpha'<\alpha} \{T_\gamma^{(n)} < T_{\alpha'}^{(m)}\} \in \mathscr{H}(n, \gamma).\end{aligned}$$

This completes the proof.

For the rest of this section let m and $\alpha > 0$ be fixed. For $n \geqq 0$ and $\gamma < \pi$ let $R_\gamma^{(n)} = \min\{T_\gamma^{(m+n)}, T_\alpha^{(m)}\}$. Note that for each μ there exists $\beta < \pi$ such that $P^\mu(R_\beta^{(n)} < T_\alpha^{(m)}) = {}^*P^\mu(R_\beta^{(n)} < T_\alpha^{(m)}) = 0$. Let $\mathscr{H}_\infty(m, \alpha)$ be the σ-field generated by sets of the form $\{X(R_\gamma^{(n)}) \in B, R_\gamma^{(n)} < \infty\}$ or $\{X(T_\alpha^{(m)}) \in B, T_\alpha^{(m)} < \infty\}$, where $B \in \mathscr{B}_\Delta$ and $\gamma < \pi$. Let $\mathscr{H}_\infty(m, \alpha)$ be the smallest σ-field containing all $\mathscr{H}_n(m, \alpha)$, $n \geqq 0$. Note that $\mathscr{H}_\infty(m, \alpha) \subset \mathscr{F}(T_\alpha^{(m)})$.

COROLLARY 3.2. $\mathscr{H}_n(m, \alpha) \cap \{T_\alpha^{(m)} = T_\gamma^{(m+n)} < \infty\} \subset \mathscr{H}(m+n, \gamma)$.

Proof. If $\Lambda = \{X(T_\alpha^{(m)}) \in B, T_\alpha^{(m)} < \infty\}$ then

$$\Lambda \cap \{T_\alpha^{(m)} = T_\gamma^{(m+n)} < \infty\} = \{X(T_\gamma^{(m+n)}) \in B, T_\alpha^{(m)} = T_\gamma^{(m+n)} < \infty\} \in \mathscr{H}(m+n, \gamma)$$

by Lemma 3.1. Let $\Lambda = \{X(R_{\gamma'}^{(n)}) \in B, R_{\gamma'}^{(n)} < \infty\}$. If $\gamma' \leqq \gamma$ then

$$\Lambda \cap \{T_\alpha^{(m)} = T_\gamma^{(m+n)} < \infty\} = \{X(T_\gamma^{(m+n)}) \in B, T_\alpha^{(m)} = T_\gamma^{(m+n)} < \infty\} \in \mathscr{H}(m+n, \gamma).$$

If $\gamma < \gamma'$, then

$$\Lambda \cap \{T_\alpha^{(m)} = T_\gamma^{(m+n)} < \infty\} = \{X(T_{\gamma'}^{(m+n)}) \in B, T_\alpha^{(m)} = T_\gamma^{(m+n)} < \infty\} \in \mathscr{H}(m+n, \gamma).$$

The corollary follows.

LEMMA 3.3. *$\mathscr{H}_n(m, \alpha)^\mu \subset \mathscr{H}_{n+1}(m, \alpha)^\mu$ for every μ and n.*

Proof. (i) The following equalities show that for arbitrary δ the sets

$$\{T_\delta^{(m+n+1)} < T_\alpha^{(m)}\} \quad \text{and} \quad \{T_\delta^{(m+n+1)} = T_\alpha^{(m)} < \infty\}$$

are in $\mathscr{H}_{n+1} = \mathscr{H}_{n+1}(m, \alpha)$:

$$\{T_\delta^{(m+n+1)} < T_\alpha^{(m)}\} = \{X(R_\delta^{(n+1)}) \neq X(R_{\delta+1}^{(n+1)}), R_{\delta+1}^{(n+1)} < \infty\} \cup \{R_\delta^{(n+1)} < \infty, R_{\delta+1}^{(n+1)} = \infty\},$$

$$\{T_\delta^{(m+n+1)} = T_\alpha^{(m)} < \infty\} = \{X(R_{\delta'}^{(n+1)}) \neq X(R_{\delta'+1}^{(n+1)}) \text{ for all } \delta' < \delta, X(R_\delta^{(n+1)}) = X(R_{\delta+1}^{(n+1)}), R_{\delta+1}^{(n+1)} < \infty\}.$$

(ii) Let us show that, if $\Lambda \in \mathscr{H}(m+n+1, \delta)$, $\Lambda \cap \{T_\delta^{(m+n+1)} < T_\alpha^{(m)}\}$ and $\Lambda \cap \{T_\delta^{(m+n+1)} = T_\alpha^{(m)} < \infty\}$ are in $\mathscr{H}_{n+1}$. We need only to consider

$$\Lambda = \{X(T_{\delta'}^{(m+n+1)}) \in B, T_{\delta'}^{(m+n+1)} < \infty\}$$

where $B \in \mathscr{B}_\Delta$, $\delta' \leqq \delta$. Now

$$\Lambda \cap \{T_\delta^{(m+n+1)} < T_\alpha^{(m)}\} = \{X(R_{\delta'}^{(n+1)}) \in B, R_{\delta'}^{(n+1)} < \infty\} \cap \{T_\delta^{(m+n+1)} < T_\alpha^{(n)}\},$$

$$\Lambda \cap \{T_\delta^{(m+n+1)} = T_\alpha^{(m)} < \infty\} = \{X(R_{\delta'}^{(n+1)}) \in B, R_{\delta'}^{(n+1)} < \infty\} \cap \{T_\delta^{(m+n+1)} = T_\alpha^{(m)} < \infty\},$$

and it follows from (i) that they are $\mathscr{H}_{n+1}$-sets.

(iii) Suppose $\Lambda = \{X(R_\gamma^{(n)}) \in B, R_\gamma^{(n)} < \infty\}$ where $B \in \mathscr{B}_\Delta$ and $\gamma < \pi$. Then Λ is the union of

$$\Lambda_1 = \{X(T_\alpha^{(m)}) \in B, T_\alpha^{(m)} < \infty\} \cap \{T_\alpha^{(m)} \leqq T_\gamma^{(m+n)}\}$$

and

$$\Lambda_2 = \{X(T_\gamma^{(m+n)}) \in B, T_\gamma^{(m+n)} < \infty\} \cap \{T_\gamma^{(m+n)} < T_\alpha^{(m)}\}.$$

Let β be such that $P^{\mu}(T_{\beta}^{(m+n+1)}<\infty)=0$. We have

$$\Lambda_1 = \Gamma_1 \cup \bigcup_{\delta<\beta} (\{X(T_{\delta}^{(m+n+1)}) \in B,\ T_{\delta}^{(m+n+1)} < \infty\} \cap \{T_{\gamma}^{(m+n)} \geqq T_{\delta}^{(m+n+1)} < \infty\} \cap \{T_{\alpha}^{(m)} = T_{\delta}^{(m+n+1)} < \infty\}),$$

$$\Lambda_2 = \Gamma_2 \cup \bigcup_{\delta<\beta} (\{X(T_{\delta}^{(m+n+1)}) \in B,\ T_{\delta}^{(m+n+1)} < \infty\} \cap \{T_{\gamma}^{(m+n)} = T_{\delta}^{(m+n+1)} < \infty\} \cap \{T_{\delta}^{(m+n+1)} < T_{\alpha}^{(m)}\}),$$

where Γ_1 and Γ_2 are $\mathscr{F}$-sets contained in $\{T_{\beta}^{(m+n+1)}<\infty\}$. Now

$$\{T_{\gamma}^{(m+n)} = T_{\delta}^{(m+n+1)} < \infty\}$$

and

$$\{T_{\delta}^{(m+n+1)} \leqq T_{\gamma}^{(m+n)},\ T_{\delta}^{(m+n+1)} < \infty\} = \{T_{\gamma}^{(m+n)} = T_{\delta}^{(m+n+1)} < \infty\} \cup \{T_{\delta}^{(m+n+1)} < T_{\gamma}^{(m+n)}\}$$

are in $\mathscr{H}(m+n+1, \delta)$ (see Lemma 3.1). It follows from (ii) that Λ_1 and Λ_2 are in $\mathscr{H}_{n+1}^{\mu}$ and hence so is Λ. This proves $\mathscr{H}_n \subset \mathscr{H}_{n+1}^{\mu}$ and the lemma follows.

LEMMA 3.4. *Let K be a compact subset of E and let $R_m=\inf\{T_{\alpha}^{(m)} \mid T_{\alpha}^{(m)} \geqq T_K\}$. If $x \notin K$, then $\{R_m=T_{\alpha}^{(m)}<\sigma\} \in \mathscr{H}_{\infty}(m, \alpha)^{\varepsilon_x}$. ($\varepsilon_x$ is the unit mass at x.)*

Proof. Since $\{R_n=T_{\alpha}^{(m)}<\sigma\}=\{T_K \leqq T_{\alpha}^{(m)}<\sigma\}-\bigcup_{\alpha'<\alpha}\{T_K \leqq T_{\alpha'}^{(m)}<\sigma\}$, it suffices to show that $\{T_K \leqq T_{\alpha'}^{(m)}<\sigma\} \in \mathscr{H}_{\infty}(m, \alpha)^{\varepsilon_x}$ for all $\alpha' \leqq \alpha$. Let $T_n=\inf\{T_{\gamma}^{(n)}<\sigma \mid \text{if } X(T_{\gamma}^{(n)}) \in V_i^{(n)} \text{ then } U_i^{(n)} \cap K \neq \varnothing\}$. By Proposition 2.5 $T_n \uparrow T_K$ or σ a.e. P^x. Hence we need only to show $\{T_n \leqq T_{\alpha'}^{(m)}<\sigma\} \in \mathscr{H}_{\infty}(m, \alpha)^{\varepsilon_x}$ for $\alpha' \leqq \alpha$, $n \geqq m$. Let $n \geqq m$ and let β be such that $P^x(T_{\beta}^{(n)}<\infty)=0$. We have

$$\{T_n \leqq T_{\alpha'}^{(m)} < \sigma\} = \Gamma \cup \bigcup_{\gamma<\beta} \{T_n = T_{\gamma}^{(n)} \leqq T_{\alpha'}^{(m)} < \sigma\} \tag{3.1}$$

where Γ is an $\mathscr{F}$-set contained in $\{T_{\beta}^{(n)}<\infty\}$. Now $\{T_{\gamma}^{(n)} \leqq T_{\alpha'}^{(m)}<\sigma\}$ is the union of $\bigcup_{\gamma \leqq \delta<\beta}\{T_{\alpha'}^{(m)}=T_{\delta}^{(n)}<\sigma\}$ and a subset (in $\mathscr{F}$) of $\{T_{\beta}^{(n)}<\infty\}$. Since for $\alpha'<\alpha$

$$\{T_{\alpha'}^{(m)} = T_{\delta}^{(n)} < \sigma\} = \{T_{\alpha'}^{(m)} = T_{\delta}^{(n)} < \sigma,\ T_{\delta}^{(n)} < T_{\alpha}^{(m)}\} \in \mathscr{H}(n, \delta) \cap \{T_{\delta}^{(n)} < T_{\alpha}^{(m)}\} \subset \mathscr{H}_{\infty}(m, \alpha)$$

(see (ii) of the proof of Lemma 3.3), we have $\{T_{\gamma}^{(n)} \leqq T_{\alpha'}^{(m)}<\sigma\} \in \mathscr{H}_{\infty}(m, \alpha)^{\varepsilon_x}$. Furthermore, $\{T_n=T_{\gamma}^{(n)} \leqq T_{\alpha}^{(m)},\ T_{\gamma}^{(n)}<\sigma\}$ is clearly in $\mathscr{H}_{\infty}(m, \alpha)$ by the definition of T_n. It follows that $\{T_n=T_{\gamma}^{(n)} \leqq T_{\alpha'}^{(m)}<\sigma\} \in \mathscr{H}_{\infty}(m, \alpha)^{\varepsilon_x}$. (3.1) then implies $\{T_n \leqq T_{\alpha'}^{(m)}<\sigma\} \in \mathscr{H}_{\infty}(n, \alpha)^{\varepsilon_x}$. The proof is complete.

4. **A martingale.** In this section we make an additional hypothesis: (a) both X and X^* satisfy the stronger form of quasi-left continuity, i.e., if stopping times T_n increase to T, then $X(T_n) \to X(T)$ a.e. (a.e.*) on $\{T<\infty\}$, and (b) $X(t) \to \Delta$ a.e. and a.e.* as $t \to \infty$. However, the following proposition depends only on hypothesis A.

PROPOSITION 4.1. *Given a partition $(\mathscr{U}, \mathscr{V})$ of E, there is a $\mathscr{B} \times \mathscr{B}$-measurable function $e: E \times E \to [0, 1]$ such that $H_{E-U_i}^{*}(x, B)=\int_B e(x, y) H_{E-U_i}(x, dy)$ if $x \in V_i$.*

Proof. For $x \in E$, $B \in \mathscr{B}$ let $q(x, B) = H_{E-U_i}(x, B)$ and $q^*(x, B) = H^*_{E-U_i}(x, B)$, where i is such that $x \in V_i$. For each B, $q(x, B)$ and $q^*(x, B)$ are $\mathscr{B}$-measurable in x. (This seems to be a standard fact; however, it is a special case of Proposition 5.1.) By hypothesis A, $q^*(x, \cdot) \leqq q(x, \cdot)$ as measures on $\mathscr{B}$ and hence $q^*(x, \cdot)$ is absolutely continuous with respect to $q(x, \cdot)$. Since $\mathscr{B}$ is countably generated, it follows from a well-known theorem that there is a real-valued function e defined on $E \times E$, measurable over $\mathscr{B} \times \mathscr{B}$, such that $q^*(x, B) = \int_B e(x, y) q(x, dy)$ for all $x \in E$, $B \in \mathscr{B}$. Obviously we may assume $0 \leqq e \leqq 1$.

Now for each n let $e_n : E \times E \to [0, 1]$ be a fixed $\mathscr{B} \times \mathscr{B}$-measurable function satisfying

$$H^*_{E-U_i^{(n)}}(x, B) = \int_B e_n(x, y) H_{E-U_i^{(n)}}(x, dy)$$

whenever $x \in V_i^{(n)}$ and $B \in \mathscr{B}$. Since $H_{E-U_i^{(n)}}(x, U_i^{(n)}) = 0$ we may assume $e_n(x, x) = 1$ for all $x \in E$. For $n \geqq 1$ and $0 < \gamma < \pi$ let $M(n, \gamma)$ be a function on Ω defined as follows:

$$\begin{aligned} M(n, \gamma) &= 0 && \text{if } T_\gamma^{(n)} \geqq \sigma, \\ &= \prod_{\delta < \gamma} e_n(X(T_\delta^{(n)}), X(T_{\delta+1}^{(n)})) && \text{if } T_\gamma^{(n)} < \sigma, \end{aligned}$$

where the infinite product $\prod_{\delta<\gamma} e_n(X(T_\delta^{(n)}), X(T_{\delta+1}^{(n)}))$ denotes the infimum of all finite products $e_n(X(T_{\delta_1}^{(n)}), X(T_{\delta_1+1}^{(n)})) \cdots e_n(X(T_{\delta_k}^{(n)}), X(T_{\delta_k+1}^{(n)}))$, $\delta_1 < \cdots < \delta_k < \gamma$. Clearly $M(n, \gamma)$ is measurable over $\mathscr{H}(n, \gamma)$, and, if $\delta < \gamma$, $M(n, \gamma) = M(n, \delta) \times [M(n, \delta')(\theta_{T_\delta^{(n)}})]$ on $\{T_\delta^{(n)} < \sigma\}$, where $\delta + \delta' = \gamma$ (i.e., γ is the δ'th ordinal after δ).

Proposition 4.2. *For every $x \in E$ and for every $\Lambda \in \mathscr{H}(n, \gamma) \cap \{T_\gamma^{(n)} < \sigma\}$*

$$\text{(4.1)} \qquad {}^*P^x(\Lambda) = E^x\{M(n, \gamma); \Lambda\}.$$

Proof. (i) We first prove (4.1) for Λ of the form $\{X(T_0^{(n)}) \in B_0, X(T_\gamma^{(n)}) \in B_1, T_\gamma^{(n)} < \sigma\}$. If $x \notin B_0$, both sides of (4.1) are zero. Hence we assume $x \in B_0$, in which case it suffices to let $\Lambda = \{X(T_\gamma^{(n)}) \in B_1, T_\gamma^{(n)} < \infty\}$. Let us define for $x \in E$, $0 < \delta < \pi$ measures $\nu_{x,\delta}$ and $\nu^*_{x,\delta}$ on $\mathscr{B}$ by setting $\nu_{x,\delta}(B) = E^x\{M(n, \delta); X(T_\delta^{(n)}) \in B, T_\delta^{(n)} < \sigma\}$ and $\nu^*_{x,\delta}(B) = {}^*P^x(X(T_\delta^{(n)}) \in B, T_\delta^{(n)} < \sigma)$. We show inductively $\nu_{x,\delta} = \nu^*_{x,\delta}$ for all x and δ. For $\delta = 1$ we have, from the previous proposition,

$$\begin{aligned} \nu^*_{x,1}(B) &= H^*_{E-U_i^{(n)}}(x, B) = \int_B e_n(x, y) H_{E-U_i^{(n)}}(x, dy) \\ &= E^x(e_n(X(T_0^{(n)}), X(T_1^{(n)})); X(T_1^{(n)}) \in B, T_1^{(n)} < \sigma) = \nu_{x,1}(B), \end{aligned}$$

i being such that $x \in V_i^{(n)}$.

(ii) Suppose $\nu_{y,\delta'} = \nu^*_{y,\delta'}$ for all $y \in E$ and $0 < \delta' < \delta$. If δ has a predecessor $\delta' > 0$, then

$$\begin{aligned} \nu^*_{x,\delta}(B) &= {}^*E^x\{{}^*P^{X(T_{\delta'}^{(n)})}(X(T_1^{(n)}) \in B, T_1^{(n)} < \sigma); T_1^{(n)} < \sigma\} \\ &= \int \nu^*_{y,1}(B) \nu^*_{x,\delta'}(dy) = \int \nu_{y,1}(B) \nu_{x,\delta'}(dy) \\ &= E^x\{M(n, \delta') E^{X(T_{\delta'}^{(n)})}[M(n, 1); X(T_1^{(n)}) \in B, T_1^{(n)} < \sigma]; T_{\delta'}^{(n)} < \sigma\} \\ &= E^x\{M(n, \delta')[M(n, 1)(\theta_{T_{\delta'}^{(n)}})]; X(T_\delta^{(n)}) \in B, T_\delta^{(n)} < \sigma\} = \nu_{x,\delta}(B). \end{aligned}$$

If δ has no predecessor, let $\delta_1 < \cdots < \delta_j \uparrow \delta$. Then $T_{\delta_j}^{(n)} \uparrow T_\delta^{(n)}$ as $j \to \infty$. Hence $X(T_{\delta_j}^{(n)}) \to X(T_\delta^{(n)})$ a.e. and a.e.* on $\{T_\delta^{(n)} < \sigma\}$. Furthermore, the additional hypothesis implies that $X(T_{\delta_j}^{(n)}) \to \Delta$ a.e. and a.e.* on $\{T_{\delta_j}^{(n)} < \sigma$ for all j, $T_\delta^{(n)} = \sigma\}$ as $j \to \infty$. Hence for $f \in \mathscr{C}_0$ we have by the bounded convergence theorem, as $j \to \infty$,

$$\int f\, d\nu^*_{x,\delta_j} = {}^*E^x\{f(X(T_{\delta_j}^{(n)})); T_{\delta_j}^{(n)} < \sigma\} \to {}^*E^x\{f(X(T_\delta^{(n)})); T_\delta^{(n)} < \sigma\} = \int f\, d\nu^*_{x,\delta}$$

$$\int f\, d\nu_{x,\delta_j} = E^x\{f(X(T_{\delta_j}^{(n)}))M(n, \delta_j); T_{\delta_j}^{(n)} < \sigma\}$$

$$\to E^x\{f(X(T_\delta^{(n)}))M(n, \delta); T_\delta^{(n)} < \sigma\} = \int f\, d\nu_{x,\delta},$$

since $M(n, \delta_j) \downarrow M(n, \delta)$ on $\{T_\delta^{(n)} < \sigma\}$. The condition $\nu^*_{x,\delta_j} = \nu_{x,\delta_j}$ for all j then implies $\int f\, d\nu^*_{x,\delta} = \int f\, d\nu_{x,\delta}$ for all $f \in \mathscr{C}_0$. Hence $\nu^*_{x,\delta} = \nu_{x,\delta}$. This completes the proof that (4.1) holds for Λ of the form $\{X(T_\gamma^{(n)}) \in B, T_\gamma^{(n)} < \sigma\}$.

(iii) We now prove (4.1) for $\Lambda = \{X(T_{\gamma_j}^{(n)}) \in B_j, j = 1, \ldots, k; X(T_\gamma^{(n)}) \in B, T_\gamma^{(n)} < \sigma\}$ where $\gamma_1 < \cdots < \gamma_k < \gamma$ and $B, B_j \in \mathscr{B}$. We may assume $\gamma_1 > 0$ for otherwise (4.1) is trivial or reduces to this case. The proof is an induction on k. Note that the case $k = 0$ is established already. Thus we assume $k > 0$ and (4.1) holds when γ is replaced by any $\delta > 0$ and Λ is replaced by any set of the form $\{X(T_{\delta_j}^{(n)}) \in A_j, j = 1, \ldots, k-1;$ $X(T_\delta^{(n)}) \in A, T_\delta^{(n)} < \sigma\}$, where $0 < \delta_1 < \cdots < \delta_{k-1} < \delta$ and $A, A_j \in \mathscr{B}$. Now with δ_j, $j = 1, \ldots, k-1$, and δ being such that $\gamma_1 + \delta_j = \gamma_{1+j}$ and $\gamma_1 + \delta = \gamma$, and with $\Lambda' = \{X(T_{\delta_j}^{(n)}) \in B_{j+1}, j = 1, \ldots, k-1; X(T_\delta^{(n)}) \in B, T_\delta^{(n)} < \sigma\}$, we have

$$\begin{aligned} {}^*P^x(\Lambda) &= {}^*E^x\{{}^*P^{X(T_{\gamma_1}^{(n)})}(\Lambda'); X(T_{\gamma_1}^{(n)}) \in B_1, T_{\gamma_1}^{(n)} < \sigma\} \\ &= \int_{B_1} \nu^*_{x,\gamma_1}(dy)\, {}^*P^y(\Lambda') = \int_{B_1} \nu_{x,\gamma_1}(dy) E^y\{M(n, \delta); \Lambda'\} \\ &= E^x\{M(n, \gamma_1) E^{X(T_{\gamma_1}^{(n)})}[M(n, \delta); \Lambda']; X(T_{\gamma_1}^{(n)}) \in B_1, T_{\gamma_1}^{(n)} < \sigma\} \\ &= E^x\{M(n, \gamma_1)[M(n, \delta)(\theta_{T_{\gamma_1}^{(n)}})]; X(T_{\gamma_j}^{(n)}) \in B_j, j = 1, \ldots, k, \\ &\qquad\qquad X(T_\gamma^{(n)}) \in B, T_\gamma^{(n)} < \sigma\} \\ &= E^x\{M(n, \gamma); \Lambda\}. \end{aligned}$$

Hence we have proved (4.1) for a big enough class of sets to guarantee that (4.1) holds for all $\Lambda \in \mathscr{H}(n, \gamma) \cap \{T_\gamma^{(n)} < \sigma\}$.

For each m and $\alpha > 0$ we define a sequence $\{M^{(n)}(m, \alpha), n = 0, 1, 2, \ldots\}$ of functions on Ω by setting

$$M^{(n)}(m, \alpha) = M(m+n, \lambda(m, \alpha, m+n)).$$

Recall that $\lambda(m, \alpha, m+n) = \inf\{\gamma \mid T_\gamma^{(m+n)} = T_\alpha^{(m)}\}$. Note that $M^{(n)}(m, \alpha) = 0$ on $\{T_\alpha^{(m)} \geqq \sigma\}$.

LEMMA 4.3. *$M^{(n)}(m, \alpha)$ is measurable over $\mathscr{H}_n(m, \alpha)^\mu$ for every μ.*

Proof. $M^{(n)}(m, \alpha) = M(m+n, \gamma)$ on $\Lambda_\gamma = \{T_\alpha^{(m)} = T_\gamma^{(m+n)} < \infty\}$. Now

$$\Lambda_\gamma = \{R_{\gamma+1}^{(n)} < \infty, X(R_{\gamma+1}^{(n)}) = X(R_\gamma^{(n)}) \neq X(R_\delta^{(n)}) \text{ for all } \delta < \gamma\} \in \mathscr{H}_n(m, \alpha).$$

Hence $H(m+n,\gamma) \cap \Lambda_\gamma \in \mathscr{H}_n(m,\alpha)$. Since $M(m+n,\gamma)$ is $\mathscr{H}(m+n,\gamma)$-measurable, its restriction to Λ_γ is measurable over $\mathscr{H}(m+n,\gamma) \cap \Lambda_\gamma$. It follows that the function $\sum_{\gamma\leqq\beta} M^{(n)}(m,\alpha)I_{\Lambda_\gamma}$ is $\mathscr{H}_n(m,\alpha)$-measurable for every β. (I_Λ is the indicator function of Λ.) But this function differs from $M^{(n)}(m,\alpha)$ on a subset of $\{R_\beta^{(n)} < T_\alpha^{(m)}\}$. Hence the lemma follows from the fact that there exists β such that $\{R_\beta^{(n)} < T_\alpha^{(m)}\}$ is a P^μ-null set.

COROLLARY 4.4. *$M^{(n)}(m,\alpha)$ is measurable over $\mathscr{F}(T_\alpha^{(m)})$.*

Proof. This follows from the previous lemma since $\mathscr{H}_n(m,\alpha) \subset \mathscr{F}(T_\alpha^{(m)})$ and $\mathscr{F}(T_\alpha^{(m)}) = \bigcap_\mu \mathscr{F}(T_\alpha^{(m)})^\mu$.

LEMMA 4.5. *For every $x \in E$ and $\Lambda \in \mathscr{H}_n(m,\alpha) \cap \{T_\alpha^{(m)} < \sigma\}$,*

$$*P^x(\Lambda) = E^x\{M^{(n)}(m,\alpha);\Lambda\}.$$

Proof. Let $\beta < \pi$ be such that $P^x(R_\beta^{(n)} < T_\alpha^{(m)}) = *P^x(R_\beta^{(n)} < T_\alpha^{(m)}) = 0$. Then Λ is the disjoint union of the sets $\Lambda \cap \{T_\alpha^{(m)} = T_\gamma^{(m+n)} < \infty\}$, $\gamma \leqq \beta$, and a set Γ with $P^x(\Gamma) = *P^x(\Gamma) = 0$. Hence

$$*P^x(\Lambda) = \sum_{\gamma\leqq\beta} *P^x(\Lambda \cap \{T_\alpha^{(m)} = T_\gamma^{(m+n)} < \infty\})$$

and

$$E^x\{M^{(n)}(m,\alpha);\Lambda\} = \sum_{\gamma\leqq\beta} E^x\{M^{(n)}(m,\alpha);\Lambda \cap [T_\alpha^{(m)} = T_\gamma^{(m+n)} < \infty]\}.$$

Now $\Lambda \cap \{T_\alpha^{(m)} = T_\gamma^{(m+n)} < \infty\} \in \mathscr{H}(m+n,\gamma)$ by Corollary 3.2, and on

$$\Lambda \cap \{T_\alpha^{(m)} = T_\gamma^{(m+n)} < \infty\} \subset \{T_\alpha^{(m)} = T_\gamma^{(m+n)} < \sigma\},$$

$M^{(n)}(m,\alpha) = M(m+n,\gamma)$ since $\lambda(m,\alpha,m+n) = \gamma$. Thus from Proposition 4.2 we have $*P^x(\Lambda \cap \{T_\alpha^{(m)} = T_\gamma^{(m+n)} < \infty\}) = E^x\{M^{(n)}(m,\alpha);\ \Lambda \cap [T_\alpha^{(m)} = T_\gamma^{(m+n)} < \infty]\}$. It follows that $*P^x(\Lambda) = E^x\{M^{(n)}(m,\alpha);\Lambda\}$.

We now prove the basic

THEOREM 4.6. *For every x, the sequence of functions $M^{(n)}(m,\alpha)$, $n = 0, 1, 2, \ldots$, is a martingale with respect to the σ-fields $\mathscr{H}_n(m,\alpha)^{\varepsilon_x}$, $n = 0, 1, 2, \ldots$, and the measure P^x.*

Proof. We have seen that $M^{(n)} = M^{(n)}(m,\alpha)$ is measurable over $\mathscr{H}_n(m,\alpha)^{\varepsilon_x}$ and $\mathscr{H}_n(m,\alpha)^{\varepsilon_x} \subset \mathscr{H}_{n+1}(m,\alpha)^{\varepsilon_x}$. Trivially $E^x\{M^{(n)}\} < \infty$ for all n. It remains to show that, for every $\Lambda \in \mathscr{H}_n(m,\alpha)$, $E^x\{M^{(n)};\Lambda\} = E^x\{M^{(n+1)};\Lambda\}$. Since $M^{(n)}(m,\alpha) = 0$ on $\{T_\alpha^{(m)} \geqq \sigma\}$, we may assume $\Lambda \subset \{T_\alpha^{(m)} < \sigma\}$. In view of Lemma 3.3 there exist Λ_1 and Λ_2 in $\mathscr{H}_{n+1}(m,\alpha)$ such that $\Lambda_1 \subset \Lambda \subset \Lambda_2 \subset \{T_\alpha^{(m)} < \sigma\}$ and $P^x(\Lambda_1) = P^x(\Lambda_2)$. By Lemma 4.5 we have

$$\begin{aligned} E^x\{M^{(n)};\Lambda\} &= *P^x(\Lambda) \leqq *P^x(\Lambda_2) = E^x\{M^{(n+1)};\Lambda_2\} = E^x\{M^{(n+1)};\Lambda\} \\ &= E^x\{M^{(n+1)};\Lambda_1\} = *P^x(\Lambda_1) \leqq *P^x(\Lambda) = E^x\{M^{(n)};\Lambda\}. \end{aligned}$$

The theorem is proved.

For each pair (m,α), $\alpha > 0$, let $M^\infty(m,\alpha)$ (interchangeably $M_{m,\alpha}^\infty$) be defined by

$$M^\infty(m,\alpha) = \liminf_{n\to\infty} M^{(n)}(m,\alpha).$$

COROLLARY 4.7. (i) *For every* x, $M^{(n)}(m,\alpha)\to M^{\infty}(m,\alpha)$ *a.e.* P^x *as* $n\to\infty$; (ii) $0\leqq M^{\infty}(m,\alpha)\leqq 1$ *and* $M^{\infty}(m,\alpha)=0$ *on* $\{T_\alpha^{(m)}\geqq\sigma\}$; (iii) $M^{\infty}(m,\alpha)$ *is* $\mathscr{F}(T_\alpha^{(m)})$-*measurable*; (iv) *for* $\Lambda\in\mathscr{H}_\infty(m,\alpha)^{\varepsilon_x}\cap\{T_\alpha^{(m)}<\sigma\}$, $E^x\{M^{\infty}(m,\alpha);\Lambda\}={}^*P^x(\Lambda)$.

Proof. (i) follows from Theorem 4.6 and the martingale convergence theorem, (ii) is trivial, and (iii) is a result of Corollary 4.4. If $\Lambda\in\mathscr{H}_n(m,\alpha)^{\varepsilon_x}\cap\{T_\alpha^{(m)}<\sigma\}$, then $E^x\{M^{(n')}(m,\alpha);\Lambda\}={}^*P^x(\Lambda)$ for all $n'\geqq n$. As $n'\to\infty$ we obtain from (i) that $E^x\{M^{\infty}(m,\alpha);\Lambda\}={}^*P^x(\Lambda)$. (ii) follows from this and the fact that $\mathscr{H}_n(m,\alpha)^{\varepsilon_x}$ increases with n.

5. **Removal of the additional hypothesis in 4.** In this section we will obtain functions $M^{\infty}(m,\alpha)$ satisfying Corollary 4.7 without the additional hypothesis in the previous section.

LEMMA 5.1. *Let* ϕ *be a bounded real-valued* $\mathscr{H}(n,\gamma)$-*measurable function. Then* $E^x\{\phi;T_\gamma^{(n)}<\sigma\}$ *and* ${}^*E^x\{\phi;T_\gamma^{(n)}<\sigma\}$ *are* $\mathscr{B}_\Delta$-*measurable in* x.

Proof. Let $W(m,i)=\{y\in E\mid \rho(y,E-U_i^{(n)})<1/m\}$. Let $\bar{Q}_m=T_{W(m,i)}$ if $X(0)\in V_i^{(n)}$ and $=\infty$ if $X(0)=\Delta$. $\bar{Q}_m$ is $\mathscr{G}$-measurable; hence so is the increasing limit Q_1 of $\bar{Q}_m$ as $m\to\infty$. Since $\{X(Q_1)\in B,\ Q_1<\infty\}\in\mathscr{G}$ for $B\in B_\Delta$, it follows from the way Q_1 is defined that Q_2 defined by $Q_2=Q_1+Q_1(\theta_{Q_1})$ for $Q_1<\infty$ and $=\infty$ for $Q_1=\infty$ is $\mathscr{G}$-measurable. In fact, let $\bar{Q}'_m=\inf\{t\geqq Q_1\mid X(t)\in W(m,i)\}$ if $X(Q_1)\in V_i^{(n)}$, and $=\infty$ if otherwise; then $\bar{Q}'_m$ are $\mathscr{G}$-measurable and increase to Q_2. From the same reasoning we see that all the Q_γ, $\gamma\leqq\pi$, defined below are $\mathscr{G}$-measurable:

$$\begin{aligned} Q_{\gamma+1} &= Q_\gamma+Q_1(\theta_{Q_\gamma}) && \text{if } Q_\alpha<\infty,\\ &=\infty && \text{if otherwise,}\\ Q_\gamma &= \sup_{\delta<\gamma} Q_\delta && \text{if } \gamma \text{ is a limit ordinal.}\end{aligned}$$

Now it follows from the quasi-left continuity that $\{T_1^{(n)}<\sigma\}\subset\{Q_1<\sigma\}$ and $T_1^{(n)}=Q_1$ a.e. and a.e.* on $\{Q_1<\sigma\}$. By the definitions of $T_\alpha^{(n)}$ and Q_α we then have $T_\alpha^{(n)}=Q_\alpha$ a.e. and a.e.* on $\{T_\alpha^{(n)}<\sigma\}\cup\{Q_\alpha<\sigma\}$. The lemma immediately follows.

Let n be fixed for a while. Denote by $\boldsymbol{\gamma}$ the set of limit ordinals $\leqq\gamma$.

We will define inductively functions $\xi_\gamma^{(n)}:\Omega\to[0,1]$, γ a limit ordinal, satisfying:

(5.1)
(i) $\xi_\gamma^{(n)}$ is measurable over $\mathscr{H}(n,\gamma)$,
(ii) if $\delta+\gamma'=\gamma$ then $\xi_{\gamma'}^{(n)}(\theta_{T_\delta^{(n)}})=\xi_\gamma^{(n)}$ a.e. on $\{T_\delta^{(n)}<\sigma\}$,
(iii) for every $x\in E$ and $\Lambda\in\mathscr{H}(n,\gamma)\cap\{T_\gamma^{(n)}<\sigma\}$, ${}^*P^x(\Lambda)$ $=E^x\{M(n,\gamma);\Lambda\}$,

where $M(n,\gamma)$ is defined as follows (for every $\gamma<\pi$):

$$\begin{aligned} M(n,\gamma) &= \Big[\prod_{\delta<\gamma} e_n(X(T_\delta^{(n)}),X(T_{\delta+1}^{(n)}))\Big]\Big[\prod_{\delta\in\boldsymbol{\gamma}}\xi_\delta^{(n)}\Big], && T_\gamma^{(n)}<\sigma,\\ &=0, && T_\gamma^{(n)}\geqq\sigma,\end{aligned}$$

$\prod_{\delta\in\boldsymbol{\gamma}}\xi_\delta^{(n)}$ meaning 1 if γ is finite. If $\gamma'=\gamma+p$, p a finite ordinal, then we can obtain from (iii) of (5.1) that

$$(5.2)\quad {}^*P^x(\Lambda)=E^x\{M(n,\gamma');\Lambda\}\quad \text{for every } x\in E \text{ and } \Lambda\in\mathscr{H}(n,\gamma')\cap\{T_{\gamma'}^{(n)}<\sigma\}.$$

This is obvious from the proof of Proposition 4.2. Hence in order to define $\xi_\gamma^{(n)}$, γ a limit ordinal, we may suppose that $\xi_\delta^{(n)}$ are defined and satisfy (5.1) for $\delta \in \boldsymbol{\gamma} - \{\gamma\}$ and (5.2) holds for all $\gamma' < \gamma$. Let $\gamma_1 < \cdots < \gamma_m \uparrow \gamma$. Let $M'(n, \gamma) = \lim_m M(n, \gamma_m)$.

Lemma 5.2. *For each* x, ${}^*P^x(X(T_\gamma^{(n)}) \in dy,\ T_\gamma^{(n)} < \sigma) = E^x\{M'(n, \gamma);\ X(T_\gamma^{(n)}) \in dy,\ T_\gamma^{(n)} < \sigma\}$ *as measures on* $\mathscr{B}$.

Proof. Suppose as $m \to \infty$ $X(T_{\gamma_1}^{(n)}) \to X(T_\gamma^{(n)}) \in V_i^{(n)}$. Then there exists m (m depening on ω) such that $X(T_{\gamma'}^{(n)}) \in U_i^{(n)}$ for $\gamma_m \leqq \gamma' < \gamma$. Necessarily $X(T_{\gamma'}^{(n)}) \in V_j^{(n)} \cap U_i^{(n)}$ for some $j \neq i$ if $\gamma_m \leqq \gamma' < \gamma$. Now if $V_j^{(n)} \cap U_i^{(n)} \neq \varnothing$ and $j \neq i$, then $U_j^{(n)} \cap V_i^{(n)} = \varnothing$ by conditions (ii) and (iii) of Proposition 2.1. This means that for all large m, $T_\gamma^{(n)} = T_{V_i^{(n)}}(\theta_{T_{\gamma_m}^{(n)}}) + T_{\gamma_m}^{(n)}$ on $\{X(T_\gamma^{(n)}) \in V_i^{(n)},\ T_\gamma^{(n)} < \sigma\}$. Let

$$O_k = \{y \in U_i^{(n)} \mid 0 < \rho(y, V_i^{(n)}) \leqq 1/k\}.$$

Now for a Borel $B \subset V_i^{(n)}$ we have from the induction hypothesis

$$\begin{aligned}
&{}^*E^x\{H_{(E-U_i^{(n)})\cup V_i^{(n)}}(X(T_{\gamma_{m'}}^{(n)}), B);\ X(T_{\gamma'}^{(n)}) \in O_k \text{ for } \gamma_m \leqq \gamma' \leqq \gamma_{m'},\ T_{\gamma_{m'}}^{(n)} < \sigma\} \\
&\quad \leqq E^x\{M(n, \gamma_{m'})H_{(E-U_i^{(n)})\cup V_i^{(n)}}(X(T_{\gamma_{m'}}^{(n)}), B);\ X(T_{\gamma'}^{(n)}) \in O_k \text{ for } \gamma_m \leqq \gamma' \leqq \gamma_{m'},\ T_{\gamma_{m'}}^{(n)} < \sigma\}.
\end{aligned}$$

Let $m' \to \infty$ to obtain by the bounded convergence theorem

$$\begin{aligned}
&{}^*E^x\{H^*_{(E-U_i^{(n)})\cup V_i^{(n)}}(X(T_\gamma^{(n)}), B);\ [X(T_{\gamma'}^{(n)}) \in O_k,\ T_\gamma^{(n)} < \sigma] \cap \Lambda(m, k)\} \\
&\qquad + {}^*P^x(\{X(T_\gamma^{(n)}) \in B,\ T_\gamma^{(n)} \in B,\ T_\gamma^{(n)} < \sigma\} \cap \Lambda(m, k)) \\
&\quad \leqq E^x\{M'(n, \gamma)H_{(E-U_i^{(n)})\cup V_i^{(n)}}(X(T_\gamma^{(n)}), B);\ [X(T_\gamma^{(n)}) \in O_k,\ T_\gamma^{(n)} < \sigma] \cap \Lambda(m, k)\} \\
&\qquad + E^x\{M'(n, \gamma);\ [X(T_\gamma^{(n)}) \in B,\ T_\gamma^{(n)} < \sigma] \cap \Lambda(m, k)\}
\end{aligned}$$

where $\Lambda(m, k) = \{X(T_{\gamma'}^{(n)}) \in O_k \text{ for } \gamma_m \leqq \gamma' < \gamma,\ T_\gamma^{(n)} < \sigma\}$. As $m \to \infty$, we obtain in the limit

$$\begin{aligned}
&{}^*E^x\{H^*_{(E-U_i^{(n)})\cup V_i^{(n)}}(X(T_\gamma^{(n)}), B);\ X(T_{\gamma'}^{(n)}) \in O_k,\ \gamma_m \leqq \gamma' \leqq \gamma, \text{ for some } m,\ T_\gamma^{(n)} < \sigma\} \\
&\qquad + {}^*P^x(X(T_\gamma^{(n)}) \in B,\ T_\gamma^{(n)} < \sigma) \\
&\quad \leqq E^x\{M'(n, \gamma)H_{(E-U_i^{(n)})\cup V_i^{(n)}}(X(T_\gamma^{(n)}), B); \\
&\qquad\qquad X(T_{\gamma'}^{(n)}) \in O_k,\ \gamma_m \leqq \gamma' \leqq \gamma, \text{ for some } m,\ T_\gamma^{(n)} < \sigma\} \\
&\qquad + E^x\{M'(n, \gamma);\ X(T_\gamma^{(n)}) \in B,\ T_\gamma^{(n)} < \sigma\}.
\end{aligned}$$

Let $k \to \infty$. Then the first terms on both sides of the above inequality approach 0. Thus we have ${}^*P^x(X(T_\gamma^{(n)}) \in B,\ T_\gamma^{(n)} < \sigma) \leqq E^x\{M'(n, \gamma);\ X(T_\gamma^{(n)}) \in B,\ T_\gamma^{(n)} < \sigma\}$ for $B \subset V_i^{(n)}$. The lemma follows.

Lemma 5.3. *There exists a function* $g_\gamma \colon E \times E \to [0, 1]$, *measurable over* $\mathscr{B} \times \mathscr{B}$, *such that*

$${}^*P^x(X(T_\gamma^{(n)}) \in B,\ T_\gamma^{(n)} < \sigma) = E^x\{g_\gamma(X(0), X(T_\gamma^{(n)}))M'(n, \gamma);\ X(T_\gamma^{(n)}) \in B,\ T_\gamma^{(n)} < \sigma\}$$

for all $x \in E$, $B \in \mathscr{B}$.

Proof. It follows from Lemma 5.1 that

$$*P^x(X(T_\gamma^{(n)}) \in B, T_\gamma^{(n)} < \sigma)$$

and

$$E^x\{M'(n, \gamma);\ X(T_\gamma^{(n)}) \in B, T_\gamma^{(n)} < \sigma\}$$

are $\mathscr{B}$-measurable in x for each $B \in \mathscr{B}$.

Lemma 5.2 then implies the existence of a function $g_\gamma\colon E \times E \to [0, 1]$, measurable over $\mathscr{B} \times \mathscr{B}$, such that

$$\begin{aligned} *P^x(X(T_\gamma^{(n)}) \in B, T_\gamma^{(n)} < \sigma) &= \int_B g_\gamma(x, y) E^x\{M'(n, \gamma);\ X(T_\gamma^{(n)}) \in dy, T_\gamma^{(n)} < \sigma\} \\ &= E^x\{g_\gamma(X(0), X(T_\gamma^{(n)}))M'(n, \gamma);\ X(T_\gamma^{(n)}) \in B, T_\gamma^{(n)} < \sigma\} \\ &\qquad \text{for all } x \in E \text{ and } B \in \mathscr{B}. \end{aligned}$$

Let δ_m be such that $\gamma_m + \delta_m = \gamma$. Suppose g_{δ_m} is also defined and satisfies Lemma 5.3 with δ_m replacing γ throughout. Let $\eta_m = g_{\delta_m}(X(T_{\gamma_m}^{(n)}), X(T_\gamma^{(n)}))$ on $\{T_\gamma^{(n)} < \sigma\}$ and $=0$ elsewhere. Let $\mathscr{H}'(n, \gamma_m)$ be the σ-field generated by $\mathscr{H}(n, \gamma_m)$ and sets of the form $\{X(T_\gamma^{(n)}) \in B, T_\gamma^{(n)} < \sigma\}$.

PROPOSITION 5.4. *For every x, $\{\eta_m, m = 1, 2, \ldots\}$ is a martingale with respect to the measure $M'(n, \gamma)dP^x$ and the σ-fields $\mathscr{H}'(n, \gamma_m)$.*

Proof. It suffices to show that for $\Lambda = \Lambda_1 \cap \{X(T_\gamma^{(n)}) \in B, T_\gamma^{(n)} < \sigma\}$, where $\Lambda_1 \in \mathscr{H}(n, \gamma_m) \cap \{T_{\gamma_m}^{(n)} < \sigma\}$ and $B \in \mathscr{B}$, $E^x\{\eta_m M'(n, \gamma); \Lambda\} = *P^x(\Lambda)$. From the previous proposition and the induction hypotheses we have

$$\begin{aligned} E^x\{\eta_m M'(n, \gamma);\ \Lambda\} &= E^x\{M(n, \gamma_m) E^{X(T_{\gamma_m}^{(n)})}[g_{\delta_m}(X(0), X(T_\gamma^{(n)}))M'(n, \gamma); \\ &\qquad\qquad X(T_\gamma^{(n)}) \in B, T_\gamma^{(n)} < \sigma];\ \Lambda_1\} \\ &= *E^x\{*P^{X(T_{\gamma_m}^{(n)})}(X(T_\gamma^{(n)}) \in B, T_\gamma^{(n)} < \sigma);\ \Lambda_1\} \\ &= *P^x(\Lambda_1 \cap \{X(T_\gamma^{(n)}) \in B, T_\gamma^{(n)} < \sigma\}) = *P^x(\Lambda). \end{aligned}$$

Let $\xi_\gamma^{(n)} = \liminf_{m \to \infty} \eta_m$. Then by the martingale convergence theorem $\eta_m \to \xi_\gamma^{(n)}$ a.e. as $m \to \infty$. We claim that $\xi_\gamma^{(n)}$ satisfies (5.1). (i) of (5.1) is obvious, and (iii) is a result of the facts that $\eta_m \to \xi_\gamma^{(n)}$ a.e. and that $E^x\{\eta_m M'(n, \gamma); \Lambda\} = *P^x(\Lambda)$ for $\Lambda \in \mathscr{H}'(n, \gamma_p) \cap \{T_\gamma^{(n)} < \sigma\}, p \leqq m$. To show (ii), note that

$$\xi_{\gamma'}^{(n)}(\theta_{T_\delta^{(n)}}) = \liminf_{m \to \infty} g_{\delta'_m}(X(T_{\delta + \gamma'_m}^{(n)}), X(T_\gamma^{(n)}))$$

on $\{T_\gamma^{(n)} < \sigma\}$, where $\gamma'_1 < \cdots < \gamma'_m \uparrow \gamma'$ and $\gamma'_m + \delta'_m = \gamma'$. Let $\{\gamma''_m\}$ be the increasing sequence formed by $\{\gamma_m\}$ and $\{\delta + \gamma'_m\}$ and let $\gamma''_m + \delta''_m = \gamma$. Then Proposition 5.4 implies $g_{\delta''_m}(X(T_{\gamma''_m}^{(n)}), X(T_\gamma^{(n)}))$ converges a.e. on $\{T_\gamma^{(n)} < \sigma\}$. This proves $\xi_{\gamma'}^{(n)}(\theta_{T_\delta^{(n)}}) = \xi_\gamma^{(n)}$ a.e. on $\{T_\gamma^{(n)} < \infty\}$.

Thus by induction we have the following proposition, in which γ need not be a limit ordinal because of a previous remark.

PROPOSITION 5.5. *$M(n, \gamma)$ is $\mathcal{H}(n, \gamma)$-measurable; for each x and $\Lambda \in \mathcal{H}(n, \gamma) \cap \{T_\gamma^{(n)} < \sigma\}$, $E^x\{M(n, \gamma); \Lambda\} = {}^*P^x(\Lambda)$.* Let $M^{(n)}(m, \alpha) = M(m+n, \lambda(m, \alpha, m+n))$ for $n \geqq 0$ and

$$M^\infty(m, \alpha) = \liminf_{n\to\infty} M^{(n)}(m, \alpha).$$

Then Lemma 4.3, Corollary 4.4, Lemma 4.5, Theorem 4.6 and finally Corollary 4.7 again hold, since their proofs depend only on the properties of $M(n, \gamma)$ stated in the above proposition.

6. Properties of the functions $M^\infty(m, \alpha)$.

LEMMA 6.1. *For any two pairs (m, α) and (m', α'), $M^\infty(m, \alpha) = M^\infty(m', \alpha')$ on $\{T_\alpha^{(m)} = T_{\alpha'}^{(m')}\}$.*

Proof. Suppose $m \leqq m'$. If $T_\alpha^{(m)} = T_{\alpha'}^{(m')}$, then

$$\begin{aligned} \lambda(m', \alpha', m'+n) &= \inf\{\gamma \mid T_\gamma^{(m'+n)} = T_{\alpha'}^{(m')}\} \\ &= \inf\{\gamma \mid T_\gamma^{(m+m'-m+n)} = T_\alpha^{(m)}\} = \lambda(m, \alpha, m+(m'-m+n)). \end{aligned}$$

Hence

$$\begin{aligned} M^{(m'-m+n)}(m, \alpha) &= M(m+(m'-m+n), \lambda(m, \alpha, m+(m'-m+n))) \\ &= M(m'+n, \lambda(m', \alpha', m'+n)) \\ &= M^{(n)}(m', \alpha') \end{aligned}$$

for all n on $\{T_\alpha^{(m)} = T_{\alpha'}^{(m')} < \sigma\}$, while on $\{T_\alpha^{(m)} = T_{\alpha'}^{(m')} \geqq \sigma\}$,

$$M^{(n)}(m, \alpha) = M^{(n)}(m', \alpha') = 0.$$

The lemma then follows from the definitions of $M^\infty(m, \alpha)$ and $M^\infty(m', \alpha')$.

For $\omega \in \Omega$, let $J(\omega) = \{t \in [0, \infty] \mid t = T_\alpha^{(m)}(\omega)$ for some $(m, \alpha)\}$. Let $M^\infty(m, 0) \equiv 1$ on Ω for all m. For $\omega \in \Omega$ let $M(\cdot, \omega)$ be defined on $J(\omega)$ by

$$M(t, \omega) = M^\infty_{m,\alpha}(\omega),$$

where (m, α) is any pair satisfying $T_\alpha^{(m)}(\omega) = t$. $M(\cdot, \omega)$ is well defined because of Lemma 6.1. For any random variable T with $T(\omega) \in J(\omega)$ for all ω, the function $M(T)$, interchangeably M_T, is defined by $M(T)(\omega) = M(T(\omega), \omega)$. In particular we have $M(T_\alpha^{(m)}) \doteq M^\infty(m, \alpha)$. We will investigate some properties of $M(\cdot, \omega)$ before setting up the desired multiplicative functional.

LEMMA 6.2. *For each ω, $M(\cdot, \omega)$ is nonincreasing on $J(\omega)$.*

Proof. Let $0 < \alpha < \alpha'$. It follows from definition that $M^{(n)}(m, \alpha) \geqq M^{(n)}(m, \alpha')$. Hence $M^\infty(m, \alpha) \geqq M^\infty(m, \alpha')$. Since $M^\infty(m, 0) \equiv 1$ and $M^\infty(m, \alpha) \leqq 1$, we have that, for each ω, $M(\cdot, \omega)$ is nonincreasing on $J_m(\omega) = \{t \in [0, \infty] \mid t = T_\alpha^{(m)}(\omega)$ for some $\alpha\}$. The fact that $J_m(\omega) \uparrow J(\omega)$ implies the lemma.

LEMMA 6.3. *Let $x \in I$. Then $M(T_1^{(m)}) \to 1$ a.e. P^x as $m \to \infty$.*

Proof. Since $I=I^*$ (Proposition 1.1), x is also in I^*. Let $i(m)$ be such that $x \in V_{i(m)}^{(m)}$. Then, since $U_{i(m)}^{(m)} \downarrow \{x\}$, ${}^*P^x(T_1^{(m)}<\infty)={}^*P^x(T_{E-U_{i(m)}^{(m)}}<\infty) \uparrow 1$ as $m \to \infty$. Now by Corollary 4.7 $E^x\{M(T_1^{(m)}); T_1^{(m)}<\infty\}={}^*P^x(T_1^{(m)}<\infty)$. It then follows from the previous lemma that $M(T_1^{(m)}) \to 1$ a.e. P^x.

Lemma 6.4. *$M(T_{\alpha+\alpha'}^{(m)})=M(T_\alpha^{(m)})[M(T_{\alpha'}^{(m)})(\theta_{T_\alpha^{(m)}})]$ a.e. on $\{T_\alpha^{(m)}<\infty\}$ for arbitrary m, α and α'.*

Proof. Obviously we may assume $\alpha, \alpha'>0$. On $\{T_\alpha^{(m)}<\infty\}$, $M^{(n)}(m, \alpha+\alpha') = M^{(n)}(m, \alpha)[M^{(n)}(m, \alpha')(\theta_{T_\alpha^{(m)}})]$. If both $\lim_n M^{(n)}(m, \alpha+\alpha')$ and $\lim_n M^{(n)}(m, \alpha)$ exist, in which case they are equal to $M(T_{\alpha+\alpha'}^{(m)})$ and $M(T_\alpha^{(m)})$ respectively, then so does $\lim_n [M^{(n)}(m, \alpha')(\theta_{T_\alpha^{(m)}})]$ and it is equal to $M(T_{\alpha'}^{(m)})(\theta_{T_\alpha^{(m)}})$. Hence the lemma follows from (i) of Corollary 4.7.

Corollary 6.5. *For arbitrary pairs (m, α) and (m', α'),*

$$M(T_\alpha^{(m)}+T_{\alpha'}^{(m')}(\theta_{T_\alpha^{(m)}})) = M(T_\alpha^{(m)})[M(T_{\alpha'}^{(m')})(\theta_{T_\alpha^{(m)}})]$$

a.e. on $\{T_\alpha^{(m)}<\infty\}$.

Proof. This follows from Lemma 6.5 and the facts that, with $m''=\max\{m, m'\}$, $T_\alpha^{(m)}(\omega)=T_\gamma^{(m'')}(\omega)$ and $T_{\alpha'}^{(m')}(\theta_{T_\alpha^{(m)}}\omega)=T_{\gamma'}^{(m'')}(\theta_{T_\gamma^{(m'')}}\omega)$ (if $T_\alpha^{(m)}(\omega)<\infty$) for some γ and γ', and that for a fixed x, there is a β such that $P^x\{T_\beta^{(m'')}<\infty\}=0$.

Proposition 6.6. *$M(\cdot, \omega)$ is right continuous on $J(\omega)$ a.e. $P^x(d\omega)$ for every x.*

Proof. We show first that for every x, $M(\cdot, \omega)$ is right continuous at 0 a.e. $P^x(d\omega)$. If $x \in I$, this follows from Lemmas 6.3 and 6.4. If $x \in H$, it follows from the fact that, a.e. $P^x(d\omega)$, there exists $\delta>0$ such that $T_1^{(m)}(\omega)>\delta$ for all m. Now let (m, α) be fixed. The strong Markov property and the above fact imply that $M(T_1^{(m')})(\theta_{T_\alpha^{(m)}}) \to 1$ a.e. on $\{T_\alpha^{(m)}<\infty, T_1^{(m')}(\theta_{T_\alpha^{(m)}}) \downarrow 0 \text{ as } m' \to \infty\}$. It then follows from Corollary 6.5 that $M(\cdot, \omega)$ is right continuous at $T_\alpha^{(m)}(\omega)$ a.e. $P^x(d\omega)$ for every x. Since for a fixed x there is a β such that $P^x(T_\beta^{(m)}<\infty)=0$ for all m, the proposition follows.

7. **The multiplicative functional—when the processes have no holding points.** In this section we assume $H=H^*=\varnothing$, i.e., both processes have no holding points. For $t \in [0, \infty)$ let $\Phi_t=\{\omega \in \Omega \mid$ for every $\delta>0$ there exists (m, α) such that $t \leqq T_\alpha^{(m)}(\omega)<t+\delta$ or $X(t, \omega)=\Delta\}$. We now extend $M(\cdot, \omega)$ from $J(\omega)$ to $[0, \infty]$ by setting

$$\begin{aligned} M(t, \omega) &= \sup\{M(T_\alpha^{(m)}(\omega), \omega) \mid t \leqq T_\alpha^{(m)}(\omega)\} \quad \text{if } \omega \in \Phi_t, \\ &= \inf\{M(T_\alpha^{(m)}(\omega), \omega) \mid t > T_\alpha^{(m)}(\omega)\} \quad \text{if } \omega \in \Omega-\Phi_t. \end{aligned}$$

Lemma 6.2 guarantees that this is indeed an extension of the original $M(\cdot, \omega)$ for every ω. M is now a function on $[0, \infty] \times \Omega$. We will denote by $M(T)$(or M_T), for any $T: \Omega \to [0, \infty]$, the function defined by $M(T)(\omega)=M(T(\omega), \omega)$. Note that

$0 \leqq M \leqq 1$ and that $M(t, \omega)$ is nonincreasing in t for every ω. Moreover, if $M(\cdot, \omega)$ is right continuous on $J(\omega)$ then it is so on $[0, \infty]$. Hence Proposition 6.6 implies that $M(t)$ is right continuous in t a.e.

PROPOSITION 7.1. *$M(t)$ is measurable over $\mathscr{F}_t$ for $0 \leqq t < \infty$.*

Proof. We show first that $\Phi_t \in \mathscr{F}_t$. For an arbitrary μ let β be such that $P^\mu\{T_\beta^{(m)} < \infty\} = 0$ for all m. Then

$$\Phi_t = \{X(t) = \Delta\} \cup \left(\bigcap_{k=1}^{\infty} \bigcup_{m=1}^{\infty} \bigcup_{\alpha<\beta} \{t \leqq T_\alpha^{(m)} < t+1/k\}\right) \cup \Gamma,$$

where Γ is a subset of $\bigcup_{m=1}^{\infty} \{T_\beta^{(m)} < \infty\}$, a P^μ-null set in $\mathscr{F}$. Since

$$\bigcap_{k=1}^{\infty} \bigcup_{m=1}^{\infty} \bigcup_{\alpha<\beta} \{t \leqq T_\alpha^{(m)} < t+1/k\} \in \bigcap_{k=1}^{\infty} \mathscr{F}(t+1/k) = \mathscr{F}(t),$$

the arbitrariness of μ implies $\Phi_t \in \mathscr{F}_t$. Now define a sequence of random variables R_n as follows:

$$\begin{aligned} R_n &= \inf\{T_\gamma^{(n)} \mid T_\gamma^{(n)} \geqq t\} \quad \text{on } \Phi_t, \\ &= \sup\{T_\gamma^{(n)} \mid T_\gamma^{(n)} < t\} \quad \text{on } \Omega - \Phi_t. \end{aligned}$$

Note that inf and sup above can be replaced by min and max respectively. Clearly as $n \to \infty$, $R_n \downarrow t$ on $\Phi_t - \{X(t) = \Delta\}$ and $R_n \uparrow \sup\{T_\alpha^{(m)} \mid T_\alpha^{(m)} < t\}$ on $\Omega - \Phi_t$. It then follows that $M(R_n) \to M(t)$ as $n \to \infty$. Now for a positive integer k let $M^{(k)}(R_n) = M(R_n) I_{\{R_n < t+1/k\}}$. Then $M^{(k)}(R_n) \to M(t)$ as $n \to \infty$. We will show that $M^{(k)}(R_n)$ is $\mathscr{F}(t+1/k)$-measurable. This would imply that $M(t)$ is $\mathscr{F}(t+1/k)$-measurable for every k and thus $\mathscr{F}(t)$-measurable. To show $M^{(k)}(R_n)$ is measurable over $\mathscr{F}(t+1/k)$, again let μ be arbitrary and choose β such that $P^\mu(T_\beta^{(n)} < \infty) = 0$ for every n. For each pair (n, γ) let $\Phi_t(n, \gamma) = \Phi_t \cap \{T_\gamma^{(n)} \geqq t;\ T_\delta^{(n)} < t \text{ for all } \delta < \gamma\}$ and $\Psi_t(n, \gamma) = \{T_\gamma^{(n)} < t \leqq T_{\gamma+1}^{(n)}\} - \Phi_t$. Clearly $\Phi_t(n, \gamma)$ and $\Psi_t(n, \gamma)$ are in $\mathscr{F}(t)$, and $R_n = T_\gamma^{(n)}$ on $\Phi_t(n, \gamma) \cup \Psi_t(n, \gamma)$. Since $M(T_\gamma^{(n)})$ is $\mathscr{F}(T_\gamma^{(n)})$-measurable, this implies that $M(R_n) I_{\Phi_t(n,\gamma) \cap \{T_\gamma^{(n)} < t+1/k\}}$ and $M(R_n) I_{\Psi_t(n,\gamma)}$ are $\mathscr{F}(t+1/k)$-measurable. Thus the function

$$\sum_{\gamma<\beta} \left(M(R_n) I_{\Phi_t(n,\gamma) \cap \{T_\gamma^{(n)} < t+1/k\}} + M(R_n) I_{\Psi_t(n,\gamma)}\right)$$

is $\mathscr{F}(t+1/k)$-measurable. Since this function differs from $M^{(k)}(R_n)$ only on a subset of $\Omega - \bigcup_{\gamma<\beta} (\Phi_t(n, \gamma) \cup \Psi_t(n, \gamma)) \subset \{T_\beta^{(n)} < \infty\}$, $M^{(k)}(R_n)$ is measurable over $\mathscr{F}(t+1/k)^\mu$. μ being arbitrary, $M^{(k)}(R_n)$ is measurable over $\mathscr{F}(t+1/k)$. The proof is thus complete.

We will now show that the restriction of M to $[0, \infty) \times \Omega$, which we will still call M, is a multiplicative functional of X and has the desired property. The assumption $H = \varnothing$ implies that for every x and t, $P^x(\{X(t) \in E\} - \Phi_t) = 0$. This fact will be strongly used in the next proof.

THEOREM 7.2. *M is a multiplicative functional of X.*

Proof. We have seen that $0 \leqq M \leqq 1$, $M(t)$ is nonincreasing and right continuous in t a.e., and $M(t)$ is $\mathscr{F}(t)$-measurable. It remains to show that for arbitrary $s, t \geqq 0$ the equality

$$M(s+t) = M(t)[M(s)(\theta_t)] \tag{7.1}$$

holds a.e. Obviously we need only to show the equality holding a.e. on $\{s+t<\sigma\}$. Since $\{s+t < \sigma\} = \{X(s+t) \in E\} = \{X(s+t) \in E\} \cap \{X(t) \in E\}$,

$$P^x(\{s+t < \sigma\} - \Phi_t \cap \Phi_{s+t}) \leqq P^x(\{X(t) \in E\} - \Phi_t) + P^x(\{X(s+t) \in E\} - \Phi_{s+t}) = 0.$$

Thus it suffices to show that (7.1) holds a.e. on $\Phi_s \cap \Phi_{s+t} \cap \{s+t<\sigma\}$.

For each n define R_n as in the proof of Proposition 7.1, and define R'_n as R_n except that t is replaced by $s+t$, i.e., $R'_n = \inf\{T_\gamma^{(n)} \mid T_\gamma^{(n)} \geqq s+t\}$ on Φ_{s+t} and $=\sup\{T_\gamma^{(n)} \mid T_\gamma^{(n)} < s+t\}$ elsewhere. On $\Phi_t \cap \Phi_{s+t}$, $R_n \leqq R'_n$, although both may be infinite. Let $S_n = R'_n - R_n$ on $\{R_n < \infty\}$. Then

$$M(R'_n) = M(R_n)[M(S_n)(\theta_{R_n})]$$

a.e. on $\{R_n < \infty\} \cap \Phi_t \cap \Phi_{s+t}$ $(M(S_n)(\theta_{R_n})(\omega) = M(S_n(\omega), \theta_{R_n}\omega))$. This follows easily from Lemma 6.4. Now $M(R_n) \to M(t)$ and $M(R'_n) \to M(s+t)$ as $n \to \infty$. Since $\{R_n < \infty, t < \sigma\} \uparrow \{t < \sigma\}$ we need only to show

$$M(S_n)(\theta_{R_n}) \to M(s)(\theta_t) \quad \text{as } n \to \infty \tag{7.2}$$

a.e. on $\Phi_t \cap \Phi_{s+t} \cap \{s+t<\sigma\}$ (regard $M(S_n)(\theta_{R_n})$ as zero if $R_n = \infty$). In view of Corollary 2.4, for each n and $\omega \in \Phi_t \cap \Phi_{s+t}$ there exist $\delta_n(\omega) \leqq \delta'_n(\omega) \leqq \pi$ such that $R_n(\omega) = t + T_{\delta_n(\omega)}^{(n+1)}(\theta_t\omega)$ and $R'_n(\omega) = t + T_{\delta'_n(\omega)}^{(n+1)}(\theta_t\omega)$. For a fixed x choose β such that $P^x(T_\beta^{(n+2)} < \infty) = 0$. It is easy to see that if $T_\beta^{(n+2)}(\omega) = \infty$ then $\delta_n(\omega)$ and $\delta'_n(\omega)$ can be chosen $\leqq \beta$. Now from Lemma 6.4 and the Markov property we have

$$M(T_{\gamma'}^{(n+1)})(\theta_t) = [M(T_\gamma^{(n+1)})(\theta_t)][M(T_{\gamma''}^{(n+1)})(\theta_{T_\gamma^{(n+1)}})(\theta_t)]$$

a.e. P^x on $\{T_\gamma^{(n+1)}(\theta_t) < \infty\}$ whenever $\gamma + \gamma'' = \gamma'$. It follows that

$$M(T_{\delta'_n}^{(n+1)})(\theta_t) = [M(T_{\delta_n}^{(n+1)})(\theta_t)][M(T_{\delta''_n}^{(n+1)})(\theta_{T_\delta^{(n+1)}})(\theta_t)] \tag{7.3}$$

a.e. P^x on $\{R_n \leqq \infty\} \cap \Phi_t \cap \Phi_{s+t}$, where $\delta''_n(\omega)$ is defined by $\delta_n(\omega) + \delta''_n(\omega) = \delta'_n(\omega)$.

Now the second factor on the right side of (7.3) is exactly $M(S_n)(\theta_{R_n})$. It is clear that $T_{\delta_n}^{(n+1)}(\theta_t) = R_n - t \downarrow 0$ and $T_{\delta''_n}^{(n+1)}(\theta_t) = R_n - t \downarrow s$ on $\Phi_t \cap \Phi_{s+t} \cap \{s+t<\sigma\}$. Since by the Markov property $M(t')(\theta_t)$ is right continuous in t' a.e. P^x, $M(T_{\delta_n}^{(n+1)})(\theta_t) \to 1$ and $M(T_{\delta''_n}^{(n+1)})(\theta_t) \to M(s)(\theta_t)$ a.e. P^x on $\Phi_t \cap \Phi_{s+t} \cap \{s+t<\sigma\}$ as $n \to \infty$. Thus (7.2) follows from (7.3), and this completes the proof of (7.1) and hence of the theorem.

LEMMA 7.3. *Let K be a compact subset of E. For each m let $R_m = \inf\{T_\alpha^{(m)} | T_\alpha^{(m)} \geqq T_K\}$. If $x \notin K$, then $E^x\{M(R_m); X(R_m) \in B, R_m < \infty\} = {}^*P^x(X(R_m) \in B, R_m < \infty)$ for all $B \in \mathscr{B}_\Delta$.*

Proof. Let β be such that $P^x(T_\beta^{(m)}<\infty)={}^*P^x(T_\beta^{(m)}<\infty)=0$. Then for $B\in\mathscr{B}_\Delta$

$$E^x\{M(R_n);\ X(R_m)\in B,\ R_m<\infty\}=\sum_{\alpha<\beta}E^x\{M(T_\alpha^{(m)});\ X(T_\alpha^{(m)})\in B,\ R_m=T_\alpha^{(m)}<\infty\}$$

and

$${}^*P^x(X(R_m)\in B,\ R_m<\infty)=\sum_{\alpha<\beta}{}^*P^x(X(T_\alpha^{(m)})\in B,\ R_m=T_\alpha^{(m)}<\infty).$$

Now it follows from Lemma 3.4 that $\{X(T_\alpha^{(m)})\in B,\ R_m=T_\alpha^{(m)}<\infty\}\in\mathscr{H}_\infty(m,\alpha)^{\varepsilon_x}$. Since obviously $R_m<\sigma$ if $R_m<\infty$, we have

$$E^x\{M(T_\alpha^{(m)});\ X(T_\alpha^{(m)})\in B,\ R_m=T_\alpha^{(m)}<\infty\}={}^*P^x(X(T_\alpha^{(m)})\in B,\ R_m=T_\alpha^{(m)}<\infty)$$

by Corollary 4.7. The desired equality follows.

THEOREM 7.4. *For every $x\in E$, compact subset K of E, and $B\in\mathscr{B}$*

$$E^x\{M(T_K);\ X(T_K)\in B,\ T_K<\infty\}=H_K^*(x,B). \tag{7.4}$$

Proof. (i) Suppose first $x\notin K$. Define R_m as in the previous proof. We have shown in Proposition 2.6 that $R_m\downarrow T_K$ a.e. P^x and a.e. ${}^*P^x$. It then follows from the a.e. right continuity of $M(t)$ and the previous lemma that, for $f\in\mathscr{C}_0$,

$$\begin{aligned}{}^*E^x\{f(X(T_K));\ T_K<\infty\}&=\lim_m {}^*E^x\{f(X(R_m));\ R_m<\infty\}\\&=\lim_m E^x\{M(R_m)f(X(R_m));\ R_m<\infty\}\\&=E^x\{M(T_K)f(X(T_K));\ T_K<\infty\}.\end{aligned}$$

The arbitrariness of f implies (7.4) for all $B\in\mathscr{B}$.

(ii) Suppose now $x\in K$. By Proposition 1.2 $x\in\operatorname{reg}K$ if and only if $x\in\operatorname{reg}^*K$. Hence if $x\in\operatorname{reg}K$, $T_K=0$ a.e. P^x and a.e. ${}^*P^x$, and it follows that

$$\begin{aligned}E^x\{M(T_K);\ X(T_K)\in B,\ T_K<\infty\}={}^*P^x(X(T_K)\in B,\ T_K<\infty)&=1\quad\text{if } x\in B,\\&=0\quad\text{if } x\notin B.\end{aligned}$$

If $x\notin\operatorname{reg}K$, then $x\notin\operatorname{reg}^*K$. Let $F_n=\{y\mid\rho(x,\Delta)/2\geqq\rho(x,y)\geqq 1/n\}$. We have $T_{F_n}\downarrow 0$ a.e. P^x and a.e. ${}^*P^x$. Hence $P^x(T_{F_n}<T_K)\uparrow 1$ and ${}^*P^x(T_{F_n}<T_K)\uparrow 1$ as $n\to\infty$. Now it follows from the fact established in (i) that, for all n,

$$\begin{aligned}&{}^*E^x\{{}^*P^{X(T_{F_n})}(X(T_K)\in B,\ T_K<\infty);\ X(T_{F_n})\notin K,\ T_{F_n}<\infty\}\\&\quad=E^x\{M(T_{F_n})E^{X(T_{F_n})}[M(T_K);\ X(T_K)\in B,\ T_K<\infty];\ X(T_{F_n})\notin K,\ T_K<\infty\}.\end{aligned}$$

Clearly the left side converges to ${}^*P^x(X(T_K)\in B,\ T_K<\infty)$ as $n\to\infty$. Since on the set $\{F_n<T_K\}$, $M(T_K)=M(T_{F_n})[M(T_K)(\theta_{T_{F_n}})]$ a.e. P^x, the right side must converge to $E^x\{M(T_K);\ X(T_K)\in B,\ T_K<\infty\}$ as $n\to\infty$. Hence (7.4) follows, and the proof is complete.

8. **The multiplicative functional—when the processes have holding points.** If $H=H^*$ is nonempty, the functional defined in the previous section does not have

the desired property (in fact it is not necessarily a multiplicative functional), and we have to make a different definition. Let $R=\inf_m T_1^{(m)}=\inf\{t<\sigma \mid X(t)\neq X(0)\}$. For each t set $M(t)=\sup\{M(T_\alpha^{(n)}) \mid T_\alpha^{(n)}\geqq t\}$ on Φ_t. Note that $M(t+R(\theta_t))$ is defined for every t. We then set on $\Omega-\Phi_t$

$$\begin{aligned} M(t) &= \min\{M(t+R(\theta_t))/M(R)(\theta_t), \inf\{M(T_\alpha^{(n)}) \mid T_\alpha^{(n)} < t\}\} \\ &\qquad\qquad \text{if } \sup\{T_\alpha^{(n)} \mid T_\alpha^{(n)} < t\} \neq T_\gamma^{(m)} \quad \text{for any } (m,\gamma), \\ &= \inf\{M(T_\alpha^{(n)}) \mid T_\alpha^{(n)} < t\} \quad \text{otherwise.} \end{aligned}$$

We will show that this M (restricted to $[0,\infty)\times\Omega$) is a desired multiplicative functional. It is obvious that $M(t)$ is nonincreasing in t everywhere. If $M(\cdot,\omega)$ is right continuous on $J(\omega)$ then it is so on $[0,\infty]$. Hence $M(t)$ is right continuous a.e.

For a while let K be a fixed compact subset of H. Define stopping times R_γ and S_γ, $\gamma<\pi$, inductively as follows: $R_0=S_0=0$; $S_{\gamma+1}=R_\gamma+T_K(\theta_{R_\gamma})$ if $R_\gamma<\infty$ and $=\infty$ if otherwise; $R_\gamma=S_\gamma+R(\theta_{S_\gamma})$ if $S_\gamma<\infty$ and $=\infty$ if otherwise; $S_\gamma=\sup_{\delta<\gamma} R_\delta$ $=\sup_{\delta<\gamma} S_\delta$ for a limit ordinal γ. Obviously, $X(S_\gamma)\in K$ a.e. and a.e.*; $R_\gamma>S_\gamma$ a.e. and a.e.* on $\{S_\gamma<\infty\}$ for $\gamma>0$, so for each μ there exists β such that $P^\mu(R_\beta<\infty)$ $={}^*P^\mu(R_\beta<\infty)=0$; for each μ, $R_\gamma=$ some $T_\alpha^{(m)}$ (m and α depending on ω) a.e. and a.e.*. For simplicity we assume that for each γ and ω $R_\gamma(\omega)=T_\alpha^{(m)}(\omega)$ for some (m,α).

Let $T_n=\inf\{T_\alpha^{(n)}<\sigma \mid \text{if } X(T_\alpha^{(n)})\in V_i^{(n)} \text{ then } K\cap U_i^{(n)}\neq\varnothing\}$. Let $\mathscr{I}_n$ be the σ-field generated by sets of the form $\{X(T_\alpha^{(m)})\in B, T_\alpha^{(m)}\leqq T_n, T_\alpha^{(m)}<\sigma\}$ or $\{X(S_1)\in B, S_1<\sigma\}$. Let $\mathscr{I}_\infty$ be the smallest σ-field containing all $\mathscr{I}_n$. Let $\mathscr{K}$ be the σ-field generated by sets of the form $\{X(T_\alpha^{(m)})\in B, T_\alpha^{(m)}\leqq R_1, T_\alpha^{(m)}<\sigma\}$. Let γ be an arbitrary limit ordinal and let $\gamma_1<\cdots<\gamma_n\uparrow\gamma$. Let $\mathscr{J}_n$ be the σ-field generated by sets of the form $\{X(T_\alpha^{(m)})\in B, T_\alpha^{(m)}\leqq R_{\gamma_n}, T_\alpha^{(m)}<\sigma\}$ or $\{X(S_\gamma)\in B, S_\gamma<\infty\}$. Let $\mathscr{J}_\infty$ be the smallest σ-field containing all $\mathscr{J}_n$. Let $\mathscr{K}_\gamma$ be the σ-field generated by sets of the form $\{X(T_\alpha^{(m)})\in B, T_\alpha^{(m)}\leqq R_\gamma, T_\alpha^{(m)}<\sigma\}$. We will omit the proofs of the following two lemmas.

LEMMA 8.1. (i) *For each* μ, $\mathscr{I}_1^\mu\subset\cdots\subset\mathscr{I}_n^\mu\subset\cdots\subset\mathscr{I}_\infty^\mu\subset\mathscr{K}^\mu$; (ii) *for each* μ $M(T_n)$ *is* $\mathscr{I}_n^\mu$*-measurable*; (iii) *for every* x *and* $\Lambda\in\mathscr{I}_n\cap\{T_n<\sigma\}$, ${}^*P^x(\Lambda)=E^x\{M(T_n);\Lambda\}$; (iv) $X(T_n)$ *is* $\mathscr{I}_n^\mu$*-measurable for each* μ.

LEMMA 8.2. (i) *For each* μ, $\mathscr{J}_1^\mu\subset\cdots\subset\mathscr{J}_n^\mu\subset\cdots\subset\mathscr{J}_\infty^\mu\subset\mathscr{K}_\gamma^\mu$; (ii) *for each* μ $M(R_{\gamma_n})$ *is* $\mathscr{J}_n^\mu$*-measurable*; (iii) *for each* x *and* $\Lambda\in\mathscr{J}_n\cap\{R_{\gamma_n}<\sigma\}$, ${}^*P^x(\Lambda)$ $=E^x\{M(R_{\gamma_n});\Lambda\}$; (iv) $X(R_{\gamma_n})$ *is* $\mathscr{J}_n^\mu$*-measurable for each* μ.

Let $h\colon E\times E\to[0,1]$ be $\mathscr{B}\times\mathscr{B}$-measurable and satisfy

$$H_K^*(x,B) = \int_B h(x,y)H_K(x,dy)$$

for $x\in E$ and $B\in\mathscr{B}$. Let $\phi_n=h(X(T_n),X(S_1))$ on $\{S_1<\sigma\}$ and $=0$ elsewhere.

LEMMA 8.3. *For each* x, $\{\phi_n M(T_n), n=1,2,\ldots\}$ *is a martingale with respect to the* σ*-fields* $\mathscr{I}_n^{\varepsilon_x}$, $n=1,2,\ldots$, *and the measure* P^x.

Proof. One shows that for $\Lambda \in \mathscr{I}_n \cap \{S_1 < \sigma\}$, $^*P^x(\Lambda) = E^x\{\phi_n M(T_n); \Lambda\}$.

Let $N(S_1) = \liminf_n \phi_n M(T_n)$. Then we have

COROLLARY 8.4. *For each x and $\Lambda \in \mathscr{K} \cap \{S_1 < \sigma\}$, $E^x\{N(S_1); \Lambda\} = {}^*P^x(\Lambda)$.*

LEMMA 8.5. (i) *$^*P^x(X(R) \in B, R < \sigma) = E^x\{M(R); X(R) \in B, R < \sigma\}$ for every x and $B \in \mathscr{B}$;* (ii) *$M(R_1) = N(S_1)[M(R)(\theta_{S_1})]$ a.e. on $\{S_1 < \infty\}$.*

Proof. (i) is easy to see. Both sides of (ii) are $\mathscr{K}^{\varepsilon_x}$-measurable, and it follows from (i) and Corollary 8.4 that for $\Lambda \in \mathscr{K}$, $E^x\{M(R_1); \Lambda\} = E^x\{N(S_1)[M(R)(\theta_{S_1})]; \Lambda\} = {}^*P^x(\Lambda \cap \{R_1 < \sigma\})$.

Let $T = \inf\{t \mid X(t) \neq X(0)\}$ and for each p let $A_p = \{y \in K \mid E^y(T) > 1/p\}$. Then $A_p \uparrow K$ as $p \to \infty$. From Lemma 8.2 we obtain for $B \in \mathscr{B}$

$$^*E^x\{H^*_{A_p}(X(R_{\gamma_n}), B); R_{\gamma_n} < \sigma\} \leqq E^x\{M(R_{\gamma_n}) H_{A_p}(X(R_{\gamma_n}), B); R_{\gamma_n} < \sigma\}.$$

It is not hard to see that $^*P^x(X(R_\delta) \in A_p$ for some $\delta, \gamma_n \leqq \delta < \gamma) \to 0$ and $P^x(X(R_\delta) \in A_p$ for some $\delta, \gamma_n \leqq \delta < \gamma) \to 0$ as $n \to \infty$. Hence as $n \to \infty$ we obtain from the above inequality

$$\begin{aligned} {}^*P^x(X(S_\gamma) \in A_p \cap B, S_\gamma < \sigma) &+ {}^*E^x\{H^*_{A_p}(X(S_\gamma), B); X(S_\gamma) \notin A_p, S_\gamma < \sigma\} \\ &\leqq E^x\left\{\lim_n M(R_{\gamma_n}); X(S_\gamma) \in A_p \cap B, S_\gamma < \sigma\right\} \\ &\quad + E^x\left\{\lim_n M(R_{\gamma_n}) H_{A_p}(X(S_\gamma), B); X(S_\gamma) \notin A_p, S_\gamma < \sigma\right\}. \end{aligned}$$

As $p \to \infty$ we obtain

$$^*P^x(X(S_\gamma) \in K \cap B, S_\gamma < \sigma) \leqq E^x\left\{\lim_n M(R_{\gamma_n}); X(S_\gamma) \in K \cap B, S_\gamma < \sigma\right\}.$$

Thus we have

LEMMA 8.6. *For every x and $B \in \mathscr{B}$,*

$$^*P^x(X(S_\gamma) \in B, S_\gamma < \sigma) \leqq E^x\left\{\lim_n M(R_{\gamma_n}); X(S_\gamma) \in B, S_\gamma < \sigma\right\}.$$

It is not hard to show that both sides of the above inequality are $\mathscr{B}_\Delta$-measurable in x. Hence there exists $c_\gamma: E \times E \to [0, 1]$, $\mathscr{B} \times \mathscr{B}$-measurable, such that

$$^*P^x(X(S_\gamma) \in B, S_\gamma < \sigma) = E^x\left\{c_\gamma(X(0), X(S_\gamma)) \lim_n M(R_{\gamma_n}); X(S_\gamma) \in B, S_\gamma < \sigma\right\}$$

for all $x \in E$ and $B \in \mathscr{B}$. Let δ_n be such that $\gamma_n + \delta_n = \gamma$ and assume that c_{δ_n} are defined. Let $\psi_n = c_{\delta_n}(X(R_{\gamma_n}), X(R_\gamma))$ on $\{R_\gamma < \sigma\}$ and $=0$ elsewhere.

LEMMA 8.7. *For each x, $\{\psi_n, n = 1, 2, \ldots\}$ is a martingale with respect to $\{\mathscr{I}_n^{\varepsilon_x}, n = 1, 2, \ldots\}$ and the measure* $\lim_m M(R_{\gamma_m})\, dP^x$.

Proof. Applying Lemma 8.2 one shows that for $\Lambda \in \mathscr{I}_n \cap \{S_\gamma < \sigma\}$,

$$^*P^x(\Lambda) = E^x\left\{\psi_n \lim_m M(R_{\gamma_m}); \Lambda\right\}.$$

Let $\psi = \liminf_n \psi_n$. Let $N(S_\gamma) = \psi \lim_m M(R_{\gamma_m})$. Then we have

COROLLARY 8.8. *For* $\Lambda \in \mathscr{J}_\infty \cap \{S_\gamma < \sigma\}$, $E^x\{N(S_\gamma); \Lambda\} = {}^*P^x(\Lambda)$.

LEMMA 8.9. $M(R_\gamma) = N(S_\gamma)[M(R)(\theta_{S_\gamma})]$ *a.e. on* $\{S_\gamma < \infty\}$.

Proof. Both sides are $\mathscr{K}_\gamma^{\varepsilon_x}$-measurable for each x. One shows that, for $\Lambda \in \mathscr{K}_\gamma$,

$$E^x\{M(R_\gamma); \Lambda\} = E^x\{N(S_\gamma)[M(R)(\theta_{S_\gamma})]; \Lambda\} = {}^*P^x(\Lambda \cap \{R_\gamma < \sigma\}).$$

For each γ let $N(S_{\gamma+1}) = M(R_\gamma)[N(S_1)(\theta_{R_\gamma})]$ on $\{R_\gamma < \sigma\}$ and $=0$ elsewhere. Then it follows from Lemma 8.5 that $M(R_{\gamma+1}) = N(S_{\gamma+1})[M(R)(\theta_{S_{\gamma+1}})]$ a.e. on $\{S_{\gamma+1} < \infty\}$.

LEMMA 8.10. *For every* γ, $M(t) = N(S_\gamma)$ *a.e. on* $\{S_\gamma \leqq t < R_\gamma\}$.

Proof. This is easily deduced from Lemmas 8.5 and 8.9, the remark preceding the lemma, and the definition of $M(t)$ (note that $M(R)(\theta_t) = M(R)(\theta_{S_\gamma})$ on $\{S_\gamma \leqq t < R_\gamma\}$).

PROPOSITION 8.11. $M(t)$ *is* $\mathscr{F}(t)$*-measurable for every* t.

Proof. According to the proof of Proposition 7.1 $M(t)I_{\{X(t) \in E_\Delta - H\}}$ is $\mathscr{F}(t)$-measurable. For a given μ there exist increasing compact subsets K_n of H such that $P^\mu(X(t) \in K_n) \uparrow P^\mu(X(t) \in H)$. It follows that to prove $M(t)I_{\{X(t) \in H\}}$ is $\mathscr{F}(t)$-measurable it suffices to show $M(t)I_{\{X(t) \in H\}}$ is $\mathscr{F}(t)$-measurable for all compact subsets K of H. Construct $N(S_\gamma)$ as before. Since $N(S_\gamma)$ is clearly $\mathscr{F}(S_\gamma)$-measurable it follows from Lemma 8.10 that $M(t)I_{\{S_\gamma \leqq t < R_\gamma\}}$ is $\mathscr{F}(t)$-measurable. This implies $M(t)I_{\{X(t) \in H\}}$ is $\mathscr{F}(t)$-measurable since for a fixed μ there exists β such that $P^\mu(R_\beta < \infty) = 0$.

PROPOSITION 8.12. *For every* s *and* t, $M(s+t) = M(t)[M(s)(\theta_t)]$ *a.e.*

Proof. We need only to show that the equality holds a.e. on $\{X(t) \in K$ or $X(s+t) \in K\}$, K a compact subset of H. And it suffices to prove the equality holding a.e. on

$$\Lambda_1 = \{S_\gamma \leqq t < R_\gamma,\ S_{\gamma'} \leqq s+t < R_{\gamma'}\} \quad (\text{we may assume } \gamma < \gamma'),$$
$$\Lambda_2 = \{S_\gamma \leqq t < R_\gamma,\ X(s+t) \in I\}, \text{ and } \Lambda_3 = \{X(t) \in I,\ S_\gamma \leqq s+t < R_\gamma\}.$$

We show only $M(s+t) = M(t)[M(s)(\theta_t)]$ a.e. on Λ_1 as the rest follows from similar arguments together with the proof of Theorem 7.2. Suppose $\gamma + \delta = \gamma'$. Then

$$\begin{aligned} M(s+t) &= N(S_{\gamma'}) = M(R_\gamma)[N(S_\delta)(\theta_{R_\gamma})] = N(S_\gamma)[M(R)(\theta_{S_\gamma})][N(S_\delta)(\theta_{R_\gamma})] \\ &= N(S_\gamma)[N(S_\delta)(\theta_{S_\delta})] = M(t)[M(s)(\theta_t)] \text{ a.e.} \end{aligned}$$

These equalities are not hard to see. For the last equality note that $M(t)$ depends only on the trajectory of the path.

Thus we have proved that M is a multiplicative functional of X. The following theorem will establish its desired property.

THEOREM 8.13. *For* $x \in E$ *and compact* $K \subset E$, $E^x\{M(T_K); X(T_K) \in B, T_K < \infty\} = H_K^*(x, B)$ *for all* $B \in \mathscr{B}$.

Proof. If $K \subset I$, the theorem follows from the proof of Theorem 7.4. If $K \subset H$, it is a result of Corollary 8.4 and the fact that $M(T_K) = M(S_1) = N(S_1)$ a.e. In general we find compact $K_m \subset K \cap H$ and compact $K'_m \subset K \cap I$ such that $T_{K_m} \downarrow T_{K \cap H}$ and $T_{K'_m} \downarrow T_{K \cap I}$ a.e. P^x and a.e. $*P^x$ so that $T_{K_m \cup K'_m} \downarrow T_K$ a.e. P^x and a.e. $*P^x$ as $m \to \infty$. It then suffices to show

$$E^x\{M(T_{K_m \cup K'_m}); X(T_{K_m \cup K'_m}) \in B, T_{K_m \cup K'_m} < \infty\} = H^*_{K_m \cup K'_m}(x, B) \tag{8.1}$$

for all B. Let $T_n = \inf\{T_\alpha^{(n)} < \sigma \mid X(T_\alpha^{(n)}) \in V_i^{(n)} \Rightarrow U_i^{(n)} \cap (K_m \cup K'_m) \neq \varnothing\}$. Let N be so large that $\rho(K_m, K'_m) = \inf\{\rho(y, z) \mid y \in K_m, z \in K'_m\} > 1/2N$, and let

$$F = \{y \in E \mid \rho(y, K_m) \leqq 1/N\} \quad \text{and} \quad F' = \{y \in E \mid \rho(y, K'_m) \leqq 1/N\}.$$

If $n \geqq N$, then we have

$$\begin{aligned} &*E^x\{H^*_{K_m}(X(T_n), B); X(T_n) \in F, T_n < \sigma\} \\ &\quad + *E^x\{H^*_{K'_m}(X(T_n), B); X(T_n) \in F', T_n < \sigma\} \\ &= E^x\{M(T_n)E^{X(T_n)}\{M(T_{K_m}); X(T_{K_m}) \in B, T_{K_m} < \infty\}; X(T_n) \in F, T_n < \sigma\} \\ &\quad + E^x\{M(T_n)E^{X(T_n)}\{M(T_{K'_m}); X(T_{K'_m}) \in B, T_{K'_m} < \infty\}; X(T_n) \in F', T_n < \sigma\}. \end{aligned} \tag{8.2}$$

Now it follows from the quasi-left continuity that $P^x(T_{K_m} < T_{K'_m}, X(T_n) \in F) \to 1$ and $P^x(T_{K'_m} < T_{K_m}, X(T_n) \in F') \to 1$ as $n \to \infty$, with similar convergences holding when $*P^x$ replaces P^x. Applying the bounded convergence theorem we obtain (8.1) from (8.2). Thus the theorem is proved.

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